Rotational Motion

Rigid body:

A body with a perfectly definite and unchanging shape is called rigid body. Rigid bodies can undergo both translational and rotational motion. Motion of a rigid body can always be represented as a combination of translational motion of some point in the body and rotational motion about an axis through that point.

Angular displacement:

If a rigid body of arbitrary shape rotating about a fixed axis and if θ represents any arbitrary angle subtended by an arc of length x on the circumference of a circle of radius R, then θ (in radians) is the angular displacement of that body and is defined as

$$\theta = \frac{x}{R}$$

Average and instantaneous angular velocity:

The average velocity of the body is defined as the ratio of the angular displacement to the elapsed time. If $\theta_1 - \theta_2$ is the angular displacement between time interval t_1 and t_2, then the average velocity is

$$\overline{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

The instantaneous velocity ω is defined as the limit approached by this ratio as Δt approaches zero, that is

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Average and instantaneous angular acceleration:

If the angular velocity of a body changes, it is said to have an angular acceleration. If ω_1 and ω_2 are the instantaneous angular velocities at times t_1 and t_2, then the average angular acceleration is defined as

$$\overline{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

The instantaneous acceleration α is defined as the limit of this ratio when Δt approaches zero.

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

Rotation with constant acceleration:

In the case of rotation with constant acceleration, expressions for the angular velocity and angular coordinate can readily be derived by integration. We have

$$\frac{d\omega}{dt} = \alpha = \text{constant}$$

$$\Rightarrow d\omega = \alpha dt$$

If ω_0 is the angular velocity when $t = 0$ and ω is the angular velocity when $t = t$, then by integration, we have

$$\int_{\omega_0}^{\omega} d\omega = \alpha \int_0^t dt$$

$$\therefore \omega = \omega_0 + \alpha t \text{ .... (1)}$$

Now, since $\omega = \dfrac{d\theta}{dt}$, we get form (1)

$$d\theta = \omega_0 dt + \alpha t dt$$

If θ_0 is the angular displacement when $t = 0$ and θ is the angular displacement when $t = t$, then by integration, we get

$$\int_{\theta_0}^{\theta} d\theta = \omega_0 \int_0^t dt + \alpha \int_0^t t dt$$

$$\therefore \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \text{ .... (2)}$$

Now, we can write the angular acceleration as

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega \frac{d\omega}{d\theta}$$

$$\Rightarrow \alpha d\theta = \omega d\omega$$

Then by integration, we have

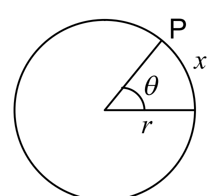
$$\alpha \int_{\theta_0}^{\theta} d\theta = \int_{\omega_0}^{\omega} \omega d\omega$$

$$\therefore \omega^2 = \omega_0{}^2 + 2\alpha(\theta - \theta_0) \text{............ (3)}$$

Equations (1) – (3) are the equations of motion with constant angular acceleration.

Relation between angular and linear velocity and acceleration:

Consider a body is moving in a circular path of radius r and at a certain time the body is at point P. If the arc x is small then the angular displacement is related with r and θ as



$$x = r\theta$$

$$\Rightarrow \Delta x = r\Delta\theta$$

$$\Rightarrow \frac{\Delta x}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$$

$$\therefore v = r\omega \text{ .... (4)}$$

This is the relation between linear and angular velocity.

Again, from (4) we can write

$$\therefore \Delta v = r\Delta\omega$$

$$\Rightarrow \frac{\Delta v}{\Delta t} = r\frac{\Delta\omega}{\Delta t}$$

$$\therefore a = r\alpha \text{ .... (5)}$$

where a is the tangential component of the linear acceleration of a point at a distance r from the axis. Equation (5) is the relation between the linear acceleration and tangential component of acceleration.

The radial component of acceleration of the point can also be expressed in terms of the angular velocity

$$a_\perp = \frac{v^2}{r} = \frac{\omega^2 r^2}{r}$$

$$\therefore a_\perp = \omega^2 r$$

Centripetal acceleration:
The word *'centripetal'* means *'seeking for a center'*. In uniform circular motion there is no tangential component of acceleration and acceleration is purely radial resulting only from the continuous change of the direction of the velocity. In this case the direction of the acceleration is towards the center and is known as centripetal acceleration which is given by

$$a_{\perp} = \frac{v^2}{r}$$

Kinetic energy of rotation:
We know the kinetic energy in linear motion is defined as

$$k = \frac{1}{2}mv^2 \text{ .... (6)}$$

In rotational motion $v = \omega r$

$$\therefore k = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}\left(mr^2\right)\omega^2$$

For a system of particles

$$k = \frac{1}{2}\left(\sum mr^2\right)\omega^2$$

$$\therefore k = \frac{1}{2}I\omega^2 \text{ .... (7)}$$

This is the expression for kinetic energy in rotational motion.

Moment of inertia:
From the definition of kinetic energy in angular motion, we get

$$I = \sum mr^2$$

The term I is known as moment of inertia. Comparing (2) with (1) it is seen that I plays a role of mass in rotational motion.
For different size and shape of the body I can have different expressions; e.g.

i) A slender rod; axis through center

$$I = \frac{1}{12}Ml^2$$

ii) Rectangular plate; axis through center

$$I = \frac{1}{12}M\left(a^2 + b^2\right)$$

iii) Hollow cylinder

$$I = \frac{1}{2}M\left(R_1^{\ 2} + R_2^{\ 2}\right)$$

iv) Solid cylinder

$$I = \frac{1}{2}MR^2$$

v) Thin walled cylinder

$$I = MR^2$$

vi) Solid sphere

$$I = \frac{2}{5} MR^2$$

Angular momentum:

The angular momentum of a moving body about an axis is defined as the product of its linear momentum and the perpendicular distance from the axis to its linear motion. It is denoted by L.

$$\therefore L = mvr$$

Comparison of linear and angular motion:

Concept	Translational motion	Rotational motion	Relationship
Displacement	x	θ	$x = r\theta$
Velocity	$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$	$v = \omega r$
Acceleration	$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$	$a = r\alpha, a_{\perp} = \frac{v^2}{r}$
Motion with constant acceleration	$v = v_0 + at$ $x = x_0 + v_0 t + \frac{1}{2}at^2$ $v^2 = {v_0}^2 + 2a(x - x_0)$	$\omega = \omega_0 + \alpha t$ $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$ $\omega^2 = {\omega_0}^2 + 2\alpha(\theta - \theta_0)$	
Mass, moment of inertia	m	I	$I = mr^2$
Kinetic energy	$K = \frac{1}{2}mv^2$	$K = \frac{1}{2}I\omega^2$	
Momentum	$P = mv$	$L = I\omega$	$L = mvr$

Problems for practice: Exercise 9-10, 9-11, 9-13, 9-14, 9-21, etc.

Exercise 9-13:

A wheel starts from rest and accelerates with constant angular acceleration to an angular velocity of $900 rev\ min^{-1}$ in $20s$. At the end of $1s$, find the angle through which the wheel has rotated.