Waves and Oscillations

Lecture No. 9

Topic: Stationary Wave (Interference by reflection)

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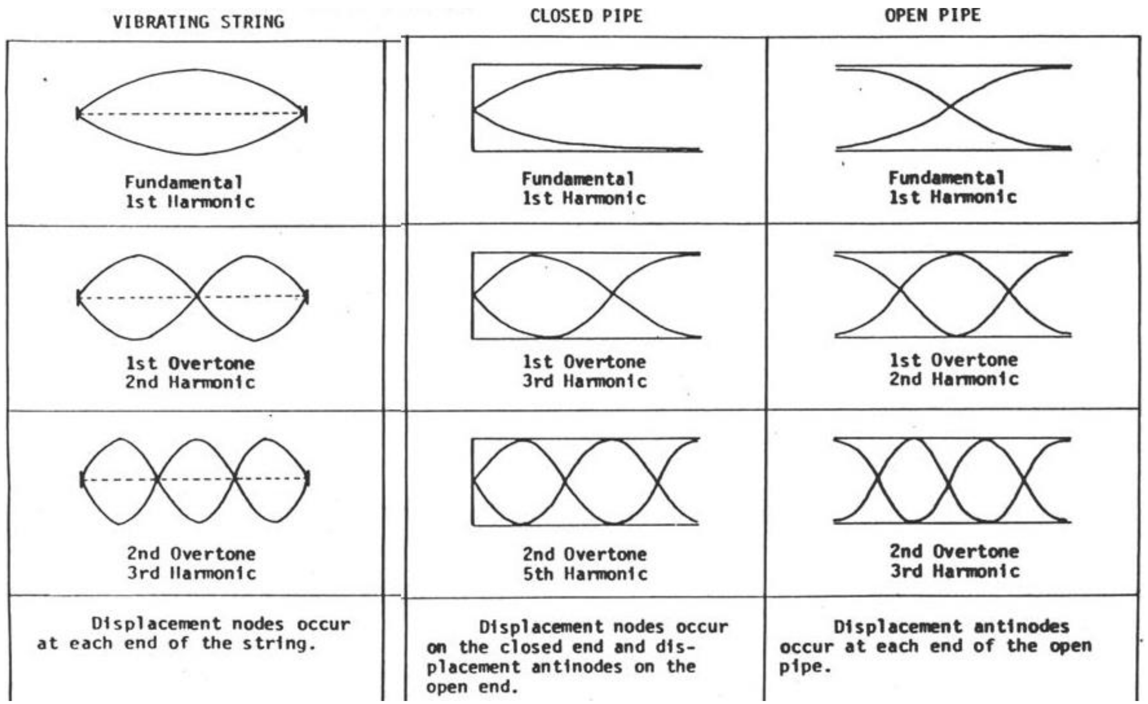
Interference by Reflection

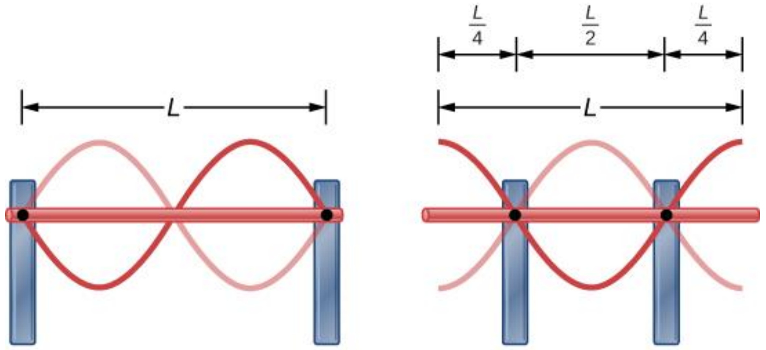
Stationary waves can be formed because of the interference caused by reflection, both in an open end system (free boundary) and closed end system (fixed boundary).

Examples:

- Open end organ pipe
- Closed end organ pipe
- Stretched string fixed at one end and free at other end
- Stretched string fixed at both ends

❑ The lowest frequency produced by any particular instrument is known as the fundamental frequency which is also known as the **1st harmonic** of the instrument.

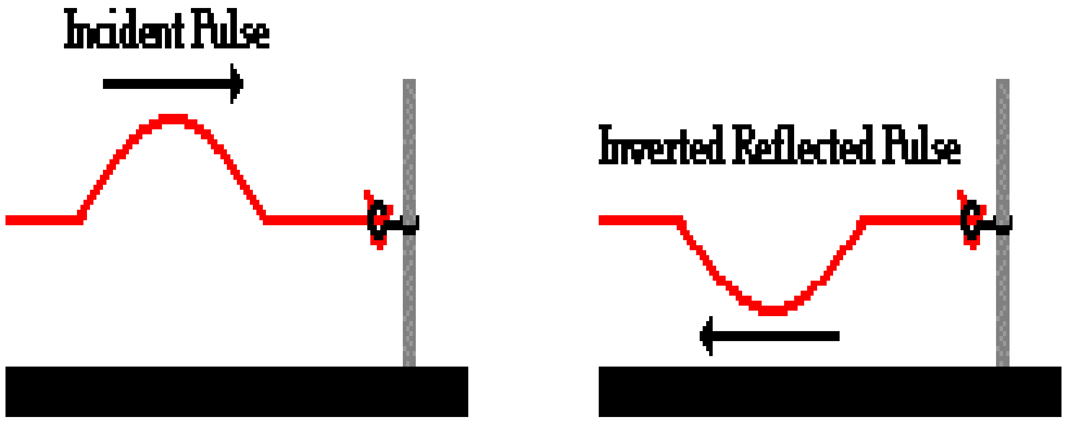


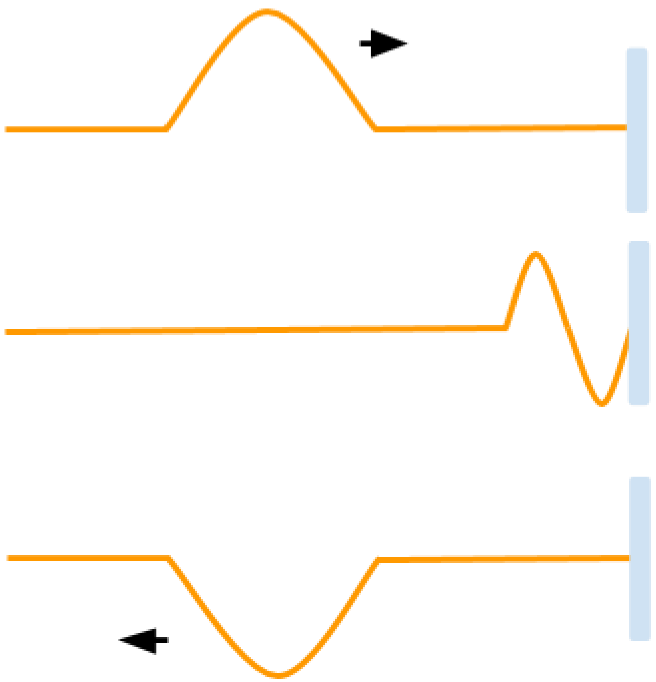


Fixed boundary conditions
$\lambda = L$

Free boundary conditions
$\lambda = L$

Reflection at a fixed end or rigid boundary





Standing wave: Reflection occurs at a fixed end or rigid boundary

The equations of a simple harmonic wave travelling along X-direction is as follows-

$y_1 = a\sin\frac{2\pi}{\lambda}(vt-x)$ (10.1)

Let this wave is incident normally on and reflected from a fixed boundary, then the equation of the reflected wave will be,

$y_2 = -a\sin\frac{2\pi}{\lambda}(vt+x)$ (10.2)

Here, both the direction of displacement of the particles and the direction of propagation of the wave itself, get reversed.

Both the incident and the reflected waves travel along the same linear path and superimposed with each other. Therefore, the equation of resultant standing wave is

$y = y_1 + y_2 = a\sin\frac{2\pi}{\lambda}(vt-x) + [-a\sin\frac{2\pi}{\lambda}(vt+x)]$

$= a[\sin\frac{2\pi}{\lambda}(vt-x) - \sin\frac{2\pi}{\lambda}(vt+x)]$

$\therefore y = a[2\cos\frac{1}{2}(\frac{2\pi}{\lambda}vt - \frac{2\pi}{\lambda}x + \frac{2\pi}{\lambda}vt + \frac{2\pi}{\lambda}x)\sin\frac{1}{2}(\frac{2\pi}{\lambda}vt - \frac{2\pi}{\lambda}x - \frac{2\pi}{\lambda}vt - \frac{2\pi}{\lambda}x)$

$= a\,2\cos(\frac{2\pi}{\lambda}vt)\sin(-\frac{2\pi}{\lambda}x)$

$y = -2a\sin\frac{2\pi x}{\lambda}\cos\frac{2\pi vt}{\lambda}$ (10.3)

$Or,\ y = -2a\sin\frac{2\pi x}{\lambda}\cos\omega t$ [since, $\omega = \frac{2\pi}{\lambda}v$]

The resultant wave is also a simple harmonic wave of the same time period and wavelength as the two constituent waves. But the amplitude is changed to,

$A = 2a\sin\frac{2\pi x}{\lambda}$ (10.4)

Velocity of the particle,

$U = \frac{dy}{dt} = \frac{4\pi a v}{\lambda}\sin\frac{2\pi x}{\lambda}\sin\frac{2\pi vt}{\lambda}$ (10.5)

Acceleration of the particle at any given instant of time,

$\frac{d^2y}{dt^2} = \frac{8\pi^2 a v^2}{\lambda^2}\sin\frac{2\pi x}{\lambda}\cos\frac{2\pi vt}{\lambda}$ (10.6)

The strain or compression at any point of the resultant vibration is given by,

$\frac{dy}{dx} = -\frac{4\pi a}{\lambda}\cos\frac{2\pi x}{\lambda}\cos\frac{2\pi vt}{\lambda}$ (10.7)

Changes with respect to position

i. Consider the points where, $sin\frac{2\pi x}{\lambda}$ = 0 and $cos\frac{2\pi x}{\lambda}$=±1

From equations (10.3), (10.4), (10.5), (10.6) and (10.7),

Displacement, *y = 0*

Amplitude, *A= 0*

Particle velocity, $\frac{dy}{dt}$ = *0*

Particle acceleration, $\frac{d^2y}{dt^2}$ = *0*

Strain, $\frac{dy}{dx} = \mp \frac{4\pi a}{\lambda} cos\frac{2\pi vt}{\lambda}$ (strain is maximum)

That means, $\frac{2\pi x}{\lambda}$ = *mπ; m=0, 1, 2, 3, …………etc.*

So, $x = \frac{m\lambda}{2} = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}$,………………etc.

These points are called nodes.

ii. Consider the points where, $sin\frac{2\pi x}{\lambda}$ = ±1 and $cos\frac{2\pi x}{\lambda}$=0

From equations (10.3), (10.4), (10.5), (10.6) and (10.7),

Displacement, $y = \mp 2a\, cos\frac{2\pi vt}{\lambda}$

Amplitude, *A= 2a*

Particle velocity, $\frac{dy}{dt} = \pm \frac{4\pi av}{\lambda} sin\frac{2\pi vt}{\lambda}$

Particle acceleration, $\frac{d^2y}{dt^2} = \pm \frac{8\pi^2 av^2}{\lambda^2} cos\frac{2\pi vt}{\lambda}$

Strain, $\frac{dy}{dx}$ = 0 (strain is zero)

That means, $\frac{2\pi x}{\lambda} = (2m+1)\frac{\pi}{2}$ *; m=0, 1, 2, 3, …………etc.*

So, $x = \frac{(2m+1)\lambda}{4} = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$………………etc.

These points are called antinodes.

Changes with respect to time

iii. Consider the instant of time, $sin\frac{2\pi vt}{\lambda}$ = 0 and $cos\frac{2\pi vt}{\lambda}$=±1

From equations (10.3), (10.4), (10.5), (10.6) and (10.7),

Displacement, $y = \mp 2a\, sin\frac{2\pi x}{\lambda}$

Amplitude, $A= 2a\, sin\frac{2\pi x}{\lambda}$ (time-independent)

Velocity, $\frac{dy}{dt} = 0$

Acceleration, $\frac{d^2y}{dt^2} = \pm\frac{8\pi^2 av^2}{\lambda^2}\, sin\frac{2\pi x}{\lambda}$

Strain, $\frac{dy}{dx} = \mp\, \frac{4\pi a}{\lambda}\, cos\frac{2\pi x}{\lambda}$ (strain is maximum)

That means, $\frac{2\pi vt}{\lambda} = m\pi$; $m=0, 1, 2, 3,$ …………*etc.*

So, $t = \frac{m\lambda}{2v} = \frac{mT}{2} = 0, \frac{T}{2}, T, \frac{3T}{2}$,………………etc. [since, $\frac{v}{\lambda} = \frac{1}{T}$]

Although at this instants amplitudes of particles are all different, each particle is at its extreme position and velocity is zero. The pattern is therefore stationary at that instant. **This instant is called the stationary instant.**

iv. Consider the instant of time, $sin\frac{2\pi vt}{\lambda}$ = ±1 and $cos\frac{2\pi vt}{\lambda}$=0

From equations (10.3), (10.4), (10.5), (10.6) and (10.7),

Displacement, $y = 0$

Amplitude, $A= 2a\, sin\frac{2\pi x}{\lambda}$ (time-independent)

Velocity, $\frac{dy}{dt} = \pm\, \frac{4\pi av}{\lambda}\, sin\frac{2\pi x}{\lambda}$

Acceleration, $\frac{d^2y}{dt^2} = 0$

Strain, $\frac{dy}{dx} = 0$

That means, $\frac{2\pi vt}{\lambda} = (2m+1)\,\frac{\pi}{2}$; $m=0, 1, 2, 3,$ …………*etc.*

So, $t = \frac{(2m+1)\lambda}{4v} = \frac{(2m+1)T}{4} = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}$……………etc. [since, $\frac{v}{\lambda} = \frac{1}{T}$]

Half a period apart, all the particles pass through their equilibrium positions and have their maximum velocities; although these maximum velocities are different for different particles.

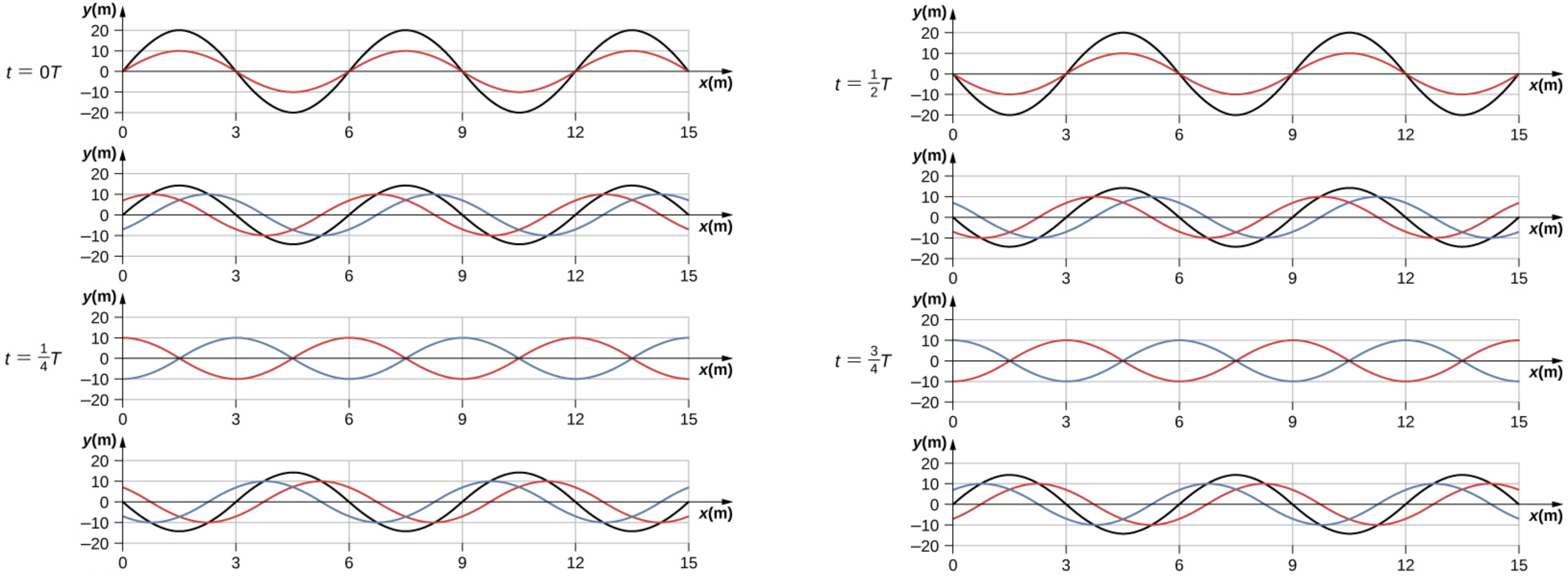
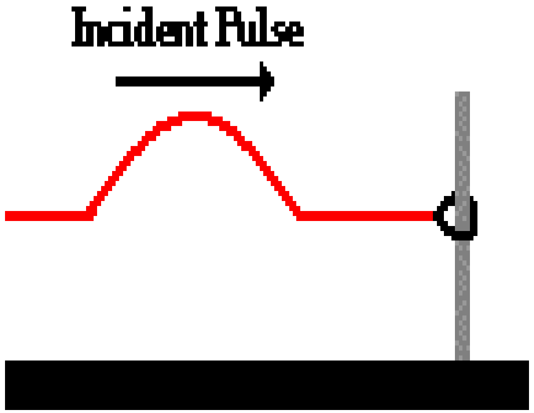
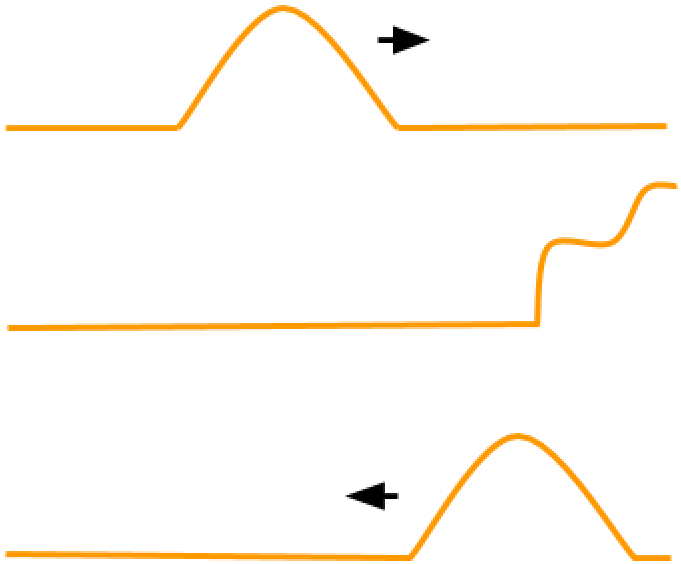


Fig. Time snapshots of two sine waves.

- The red wave is moving in the −x-direction. The blue wave is moving in the +x-direction. The resulting wave is shown in black.
- Consider the resultant wave at the points x=0m, 3m, 6m, 9m, 12m, 15m, The resultant wave always equals zero at these points, no matter what the time is. These points are known as fixed points (nodes).
- In between each two nodes is an antinode, a place where the medium oscillates with an amplitude equal to the sum of the amplitudes of the individual waves.

Reflection at a free end





Standing wave: Reflection occurs at a free end

The equations of a simple harmonic wave travelling along X-direction is as follows-

$y_1 = a \sin\frac{2\pi}{\lambda}(vt-x)$ (10.8)

Let this wave is incident normally on and reflected from a free boundary, then the equation of the reflected wave will be,

$y_2 = a \sin\frac{2\pi}{\lambda}(vt+x)$ (10.9)

Here, the direction of propagation of the wave is reversed, but the direction of displacement of the particles remains unchanged.

Both the incident and the reflected waves travel along the same linear path and superimposed with each other. Therefore, the equation of resultant standing wave is

$y = y_1 + y_2 = a \sin\frac{2\pi}{\lambda}(vt-x) + a \sin\frac{2\pi}{\lambda}(vt+x)$

$= a[2\sin\frac{1}{2}(\frac{2\pi}{\lambda}vt - \frac{2\pi}{\lambda}x + \frac{2\pi}{\lambda}vt + \frac{2\pi}{\lambda}x)\cos\frac{1}{2}(\frac{2\pi}{\lambda}vt - \frac{2\pi}{\lambda}x - \frac{2\pi}{\lambda}vt - \frac{2\pi}{\lambda}x)$

$\therefore y = a\, 2\sin(\frac{2\pi vt}{\lambda})\cos(-\frac{2\pi x}{\lambda})$

$y = 2a\cos\frac{2\pi x}{\lambda}\sin\frac{2\pi vt}{\lambda}$ (10.10)

$= 2a\cos\frac{2\pi x}{\lambda}\sin\omega t$ [since, $\omega = \frac{2\pi V}{\lambda} = 2\pi n$]

The resultant wave is also a simple harmonic wave of the same time period and wavelength as the two constituent waves. But the amplitude is changed to,

A= $2a\cos\frac{2\pi x}{\lambda}$ (10.11)

Velocity of the particle,

U= $\frac{dy}{dt} = \frac{4\pi a v}{\lambda}\cos\frac{2\pi x}{\lambda}\cos\frac{2\pi vt}{\lambda}$ (10.12)

Acceleration of the particle at any given instant of time,

$\frac{d^2y}{dt^2} = -\frac{8\pi^2 a v^2}{\lambda^2}\cos\frac{2\pi x}{\lambda}\sin\frac{2\pi vt}{\lambda}$ (10.13)

The strain or compression at any point of the resultant vibration is given by,

$\frac{dy}{dx} = -\frac{4\pi a}{\lambda}\sin\frac{2\pi x}{\lambda}\sin\frac{2\pi vt}{\lambda}$ (10.14)

Changes with respect to position

i. Consider the points where, $sin\frac{2\pi x}{\lambda}$ = 0 and $cos\frac{2\pi x}{\lambda}$=±1

From equations (10.10), (10.11), (10.12), (10.13) and (10.14),

Displacement, $y = \pm 2asin\frac{2\pi vt}{\lambda}$

Amplitude, $A= 2a$

Particle velocity, $\frac{dy}{dt} = \pm\frac{4\pi av}{\lambda}\ cos\ \frac{2\pi vt}{\lambda}$

Particle acceleration, $\frac{d^2y}{dt^2} = \mp\ \frac{8\pi^2 av^2}{\lambda^2}\ sin\frac{2\pi vt}{\lambda}$

Strain, $\frac{dy}{dx}$ = 0

That means, $\frac{2\pi x}{\lambda} = m\pi$; *m=0, 1, 2, 3, …………etc.*

So, $x = \frac{m\lambda}{2}$ = 0, $\frac{\lambda}{2}$, λ, $\frac{3\lambda}{2}$,………………etc.

These points are called antinodes.

ii. Consider the points where, $sin\frac{2\pi x}{\lambda}$ = ±1 and $cos\frac{2\pi x}{\lambda}$=0

From equations (10.10), (10.11), (10.12), (10.13) and (10.14),

Displacement, $y = 0$

Amplitude, $A= 0$

Particle velocity, $\frac{dy}{dt} = 0$

Particle acceleration, $\frac{d^2y}{dt^2} = 0$

Strain, $\frac{dy}{dx} = \mp\ \frac{4\pi a}{\lambda}\ sin\frac{2\pi vt}{\lambda}$ (strain is maximum)

That means, $\frac{2\pi x}{\lambda} = (2m+1)\ \frac{\pi}{2}$; *m=0, 1, 2, 3, …………etc.*

So, $x = \frac{(2m+1)\lambda}{4} = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$………………etc.

These points are called nodes.

Changes with respect to time

iii. Consider the instant of time, $sin\frac{2\pi vt}{\lambda}$ = 0 and $cos\frac{2\pi vt}{\lambda}$=±1

From equations (10.10), (10.11), (10.12), (10.13) and (10.14),

Displacement, $y = 0$

Amplitude, $A= 2acos\frac{2\pi x}{\lambda}$ (time-independent)

Particle velocity, $\frac{dy}{dt} = \pm\frac{4\pi av}{\lambda}\, cos\,\frac{2\pi x}{\lambda}$

Particle acceleration, $\frac{d^2y}{dt^2} = 0$

Strain, $\frac{dy}{dx} = 0$

That means, $\frac{2\pi vt}{\lambda} = m\pi$; $m=0, 1, 2, 3,$ …………etc.

So, $t = \frac{m\lambda}{2v} = \frac{mT}{2} = 0, \frac{T}{2}, T, \frac{3T}{2}$,……………….etc. [since, $\frac{v}{\lambda} = \frac{1}{T}$]

Thus, particle displacement will be zero while its velocity will be maximum at instants of time t = 0, $\frac{T}{2}$, T, $\frac{3T}{2}$,……………….etc.

iv. Consider the instant of time, $sin\frac{2\pi vt}{\lambda}$ = ±1 and $cos\frac{2\pi vt}{\lambda}$=0

From equations (10.10), (10.11), (10.12), (10.13) and (10.14),

Displacement, $y = \pm 2acos\frac{2\pi x}{\lambda}$

Amplitude, $A= 2acos\frac{2\pi x}{\lambda}$ (time-independent)

Particle velocity, $\frac{dy}{dt} =0$

Particle acceleration, $\frac{d^2y}{dt^2} = \mp\frac{8\pi^2av^2}{\lambda^2}\, cos\frac{2\pi x}{\lambda}$

Strain, $\frac{dy}{dx} = \mp\frac{4\pi a}{\lambda}\, sin\frac{2\pi x}{\lambda}$

That means, $\frac{2\pi vt}{\lambda} = (2m+1)\,\frac{\pi}{2}$; $m=0, 1, 2, 3,$ …………etc.

So, $t = \frac{(2m+1)\lambda}{4v} = \frac{(2m+1)T}{4} = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}$……………etc. [since, $\frac{v}{\lambda} = \frac{1}{T}$]

Although at this instants amplitudes of particles are all different, each particle is at its extreme position suffering the maximum strain and acceleration and velocity is zero. The pattern is therefore stationary at that instant. **This instant is called the stationary instant.**

Energy of a stationary wave

Let us consider a longitudinal wave propagates through a fluid. The bulk modulus (K) of the fluid is given by,

$K=-\frac{p}{dy/dx}$ (11.1)

Where, p is the excess pressure or volume stress and $\frac{dy}{dx}$ is the volume strain.

$\therefore p =- K\frac{dy}{dx}$ (11.2)

In the case of a stationary wave formed by reflection at a free boundary, the strain is given by,

$\frac{dy}{dx} =- \frac{4\pi a}{\lambda} sin\frac{2\pi x}{\lambda} sin\frac{2\pi vt}{\lambda}$ (11.3)

Also, phase velocity is related to bulk modulus and the density of medium (ρ) as follows,

$v=\sqrt{\frac{K}{\rho}}$ or, $v^2=\frac{K}{\rho}$ or, $K= \rho v^2$ (Newton–Laplace equation)

Substituting the values of $\frac{dy}{dx}$ and K in equation (11.2)

$p = \rho v^2[\frac{4\pi a}{\lambda} sin\frac{2\pi x}{\lambda} sin\frac{2\pi vt}{\lambda}]$ (11.4)

When, $sin\frac{2\pi x}{\lambda}=1$ and $sin\frac{2\pi vt}{\lambda}=1$, the excess pressure is the maximum.

$p_o = \rho v^2[\frac{4\pi a}{\lambda}]$ (11.5)

From equations (11.4) and (11.5),

$p = p_o sin\frac{2\pi x}{\lambda} sin\frac{2\pi vt}{\lambda}$ (11.6)

Let us put $p_o sin\frac{2\pi x}{\lambda} = p_X$ in the equation (11.6)

$p = p_X sin\frac{2\pi vt}{\lambda}$ (11.7)

The particle velocity at a point is given by,

$U=\frac{dy}{dt}=\frac{4\pi av}{\lambda} cos\frac{2\pi x}{\lambda} cos\frac{2\pi vt}{\lambda}$ (11.8)

Energy of a stationary wave

Let us put $\frac{4\pi a v}{\lambda} cos \frac{2\pi x}{\lambda}$ = U_X in the equation (11.8)

$$U=U_X cos \frac{2\pi vt}{\lambda} \qquad (11.9)$$

Now, the energy transferred per unit area in a small interval of time dt is equal to the work done,

$dI= pUdt$

So, the energy transferred during the whole time period T is given by

$I= \int_0^T pUdt$

$=\int_0^T p_X \, sin \frac{2\pi vt}{\lambda} U_X cos \frac{2\pi vt}{\lambda} dt$

Thus, the rate of energy transfer or the average energy transferred per second,

$I_{avg}= \frac{\int_0^T pUdt}{T}$

$I_{avg} = \frac{p_X U_X}{T} \int_0^T sin \frac{2\pi vt}{\lambda} cos \frac{2\pi vt}{\lambda} dt$

$= \frac{p_X U_X}{2T} \int_0^T sin \frac{4\pi vt}{\lambda} dt$

$= - \frac{p_X U_X \lambda}{2T 4\pi v} [cos \frac{4\pi vt}{\lambda}]_0^T$

$=- \frac{p_X U_X T}{8\pi T} [cos \frac{4\pi vT}{\lambda} - cos 0]$

$=- \frac{p_X U_X T}{8\pi T} [cos \frac{4\pi T}{T} - cos 0]$

$=- \frac{p_X U_X T}{8\pi T} [cos 4\pi - cos 0] =- \frac{p_X U_X T}{8\pi T} [1 - 1]$

$=0$

So, the rate of energy transfer=0

Thus, there is no transference of energy across any section of the medium in the case of a stationary or standing wave.

Sample Problems

1. The equation of a transverse wave is $y=0.004\ sin\frac{2\pi}{3}(6t\text{-}x)$ which is incident at a rigid boundary.

   i. Find the wavelength, phase velocity and time period of the incident wave?

   ii. What is the equation of the reflected wave?

   iii. Find the equation of displacement of the resultant wave.

   iv. Calculate the maximum displacement at x=2.4 m.

2. A tuning fork A of frequency 384 Hz gives 6 beats per second when sounded with another tuning fork B. On loading B with a little wax, the number of beats per second becomes 4. What is the frequency of B?

3. A sound wave in air, having an amplitude of 0.005 cm and frequency 700 Hz travelling along the direction of positive X-axis with a velocity of 350 m/s. The wave suffers reflection at a free boundary. Obtain the values of

   i. Displacement and amplitude of the resultant stationary waves at the point x=50cm.

   ii. Find out the maximum excess pressure at the point x=50cm.

   iii. Find out the distance between two successive nodes and antinodes.