Waves and Oscillations

Lecture No. 3

Topics: Combination of simple harmonic oscillations and Lissajous figures

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The Superposition of Oscillatory Motions

- Many Physical situations involves the simultaneous application of 2 or more periodic oscillations to the same system.

- Examples: A photograph stylus, a microphone diaphragm, a human eardrum, etc. are generally subjected to a complicated combinations of many periodic oscillations.

- The basic assumption in assessment of such conditions is:

"The resultant of two or more harmonic vibrations will be taken to be simply the sum of the individual vibrations."

Principle of Superposition

Suppose we have two simple harmonic motions (SHM) described by the following equations:

$$y_1 = a_1 \sin(\omega t + \alpha_1) \qquad (4.11)$$

$$y_2 = a_2 \sin(\omega t + \alpha_2) \qquad (4.12)$$

Here, y_1 and y_2 are the displacements of the particles due to the individual vibrations of amplitudes a_1 and a_2, respectively and angles of epoch α_1 and α_2, respectively. Two vibrations have the same angular frequency (ω).

The resultant displacement of the particle will be given by,

$$y = y_1 + y_2$$

$$= a_1 \sin(\omega t + \alpha_1) + a_2 \sin(\omega t + \alpha_2)$$

$$= a_1(\sin \omega t \cos \alpha_1 + \cos \omega t \sin \alpha_1) + a_2(\sin \omega t \cos \alpha_2 + \cos \omega t \sin \alpha_2)$$

$$= (a_1 \cos \alpha_1 + a_2 \cos \alpha_2) \sin \omega t + (a_1 \sin \alpha_1 + a_2 \sin \alpha_2) \cos \omega t \qquad (4.13)$$

Since a_1, a_2 (amplitudes) and $\alpha_1$and α_2 (angle of epoch) are constants we can replace them with the following constant terms:

$$a_1 \, cos \, \alpha_1 + a_2 \, cos \, \alpha_2 = A \, cos \, \varphi$$

$$a_1 \, sin \, \alpha_1 + a_2 \, sin \, \alpha_2 = A \, sin \, \varphi$$

The resultant displacement can be written as,

$$y = Acos\varphi sin\omega t + Asin\varphi cos\omega t = Asin(\omega t + \varphi) \qquad (4.14)$$

Thus, the equation (4.13) is similar to the equations (4.11) and (4.12). The resultant vibration is therefore representing a SHM with the amplitude A and epoch angle φ.

Expression of A and φ:

$$A^2sin^2\varphi + A^2cos^2\varphi = a_1^2 \, sin^2 \, \alpha_1 + a_2^2 \, sin^2 \, \alpha_2 + 2a_1a_2sin\alpha_1sin\alpha_2 + a_1^2 \, cos^2 \, \alpha_1 + a_2^2 \, cos^2 \, \alpha_2 + 2a_1a_2cos\alpha_1cos\alpha_2$$

Or, $A^2(sin^2\varphi + cos^2\varphi) = a_1^2 \, (sin^2 \, \alpha_1 + cos^2\alpha_1) + \; a_2^2(sin^2\alpha_2 + cos^2\alpha_2) + 2a_1a_2(sin\alpha_1sin\alpha_2 + cos\alpha_1cos\alpha_2)$

Or, $A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\alpha_1 - \alpha_2)$

Or, $A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos(\alpha_1 - \alpha_2)}$ (4.15)

$\tan \varphi = \frac{Asin\varphi}{Acos\varphi} = \frac{a_1 \, sin \, \alpha_1 + a_2 \, sin \, \alpha_2}{a_1 \, cos \, \alpha_1 + a_2 \, cos \, \alpha_2}$ (4.16)

Some Special Cases:

i. Same Phase: If $\alpha_1 = \alpha_2 = \alpha, (\alpha_1 - \alpha_2) = 0, 2\pi, 4\pi, \dots\dots\dots = 2n\pi;\ \ n = 0, 1, 2, 3, \dots\dots\dots\dots\dots$

Then, $\cos(\alpha_1 - \alpha_2) = 1$ and $A^2 = a_1^2 + a_2^2 + 2a_1a_2 = (a_1 + a_2)^2$; So, $A = (a_1 + a_2)$

$$tan\varphi = \frac{(a_1+a_2)sin\alpha}{(a_1+a_2)cos\alpha} = tan\alpha$$

In this case, $y = (a_1 + a_2)\sin(\omega t + \alpha)$ (4.17)

ii. Opposite Phase: If $(\alpha_1 - \alpha_2) = \pi, 3\pi, 5\pi, \dots\dots\dots = (2n + 1)\pi;\ \ n = 0, 1, 2, 3, \dots\dots\dots\dots\dots$

Then, $\cos(\alpha_1 - \alpha_2) = -1$ and $A^2 = a_1^2 + a_2^2 - 2a_1a_2 = (a_1 - a_2)^2$; So, $A = (a_1 - a_2)$

iii. If $a_1 = a_2 = a$; The same phase condition gives, $A = 2a$; Resultant amplitude is the maximum.

The opposite phase condition gives $A = 0$; Resultant amplitude is zero.

iv. If $\alpha_1 - \alpha_2 = \frac{\pi}{2};\ A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos\frac{\pi}{2}} = \sqrt{a_1^2 + a_2^2}$

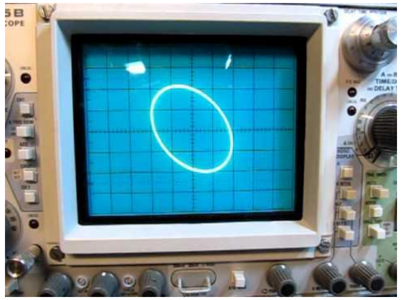
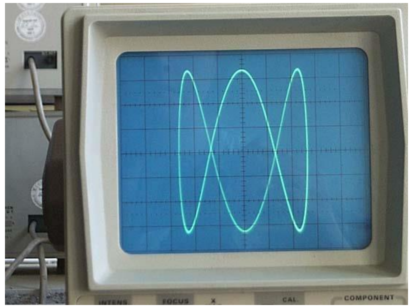
In this case if $a_1 = a_2 = a, A = \sqrt{2a^2} = \sqrt{2}a$

Lissajous Figures

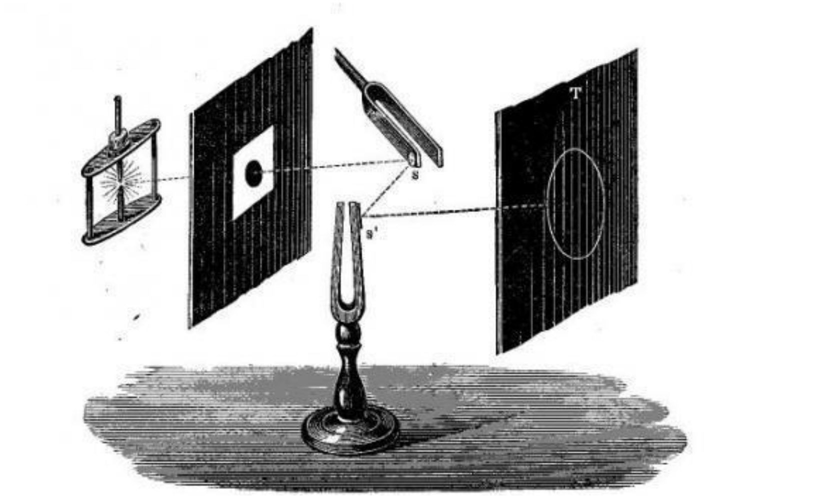
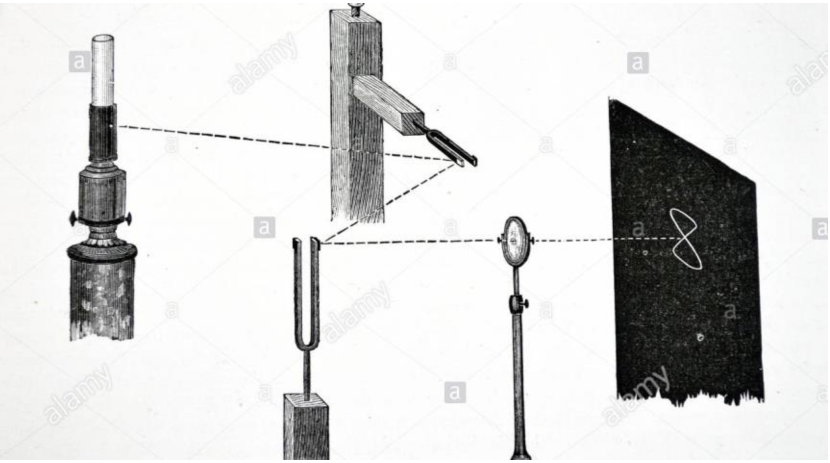
- Jules Antoine Lissajous (4 March, 1822– 24 June, 1880) was a French physicist, after whom Lissajous figures are named.

- When two SHM simultaneously act on a particle at right angle to each other, the resultant motion of the particle traces a curve. Any of an infinite variety of curves formed by combining two mutually perpendicular simple harmonic motions are called Lissajous figures.

- Commonly exhibited by the oscilloscope. Used in studying frequency, amplitude and phase relations of harmonic vibrations.

Lissajous figures on oscilloscope

Set up of Lissajous's experiment

Combination of two SHMs' at right angles to each other (frequency ratio 1:1, different phase and different amplitudes)

The equations of displacement of the SHMs' at right angles to each other are as follows,

$x = a\ sin(\omega t + \varphi)$ (5.1)

$y = b\ sin\omega t$ (5.2)

Let us relate the two equations of displacements as follows,

$\frac{x}{a} = sin(\omega t + \varphi) = sin\omega t\ cos\varphi + cos\omega t\ sin\varphi$

$= \frac{y}{b}\ cos\varphi\ + \sqrt{1 - \frac{y^2}{b^2}}\ sin\varphi$ [from equation (5.2)]

$\therefore\ \frac{x}{a} - \frac{y}{b}\ cos\varphi = \sqrt{1 - \frac{y^2}{b^2}}\ sin\varphi;$ Now, let us take square on both sides of the equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}\ cos^2\varphi - 2\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)\ cos\varphi = \left(1 - \frac{y^2}{b^2}\right)\ sin^2\varphi$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2}\ cos^2\varphi + \frac{y^2}{b^2}\ sin^2\varphi - 2\left(\frac{xy}{ab}\right) cos\varphi = sin^2\varphi$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2}\ - 2\left(\frac{xy}{ab}\right) cos\varphi = sin^2\varphi \qquad (5.3)$$

Equation (5.3) is the general equation of the resultant vibration of the two mentioned SHMs'.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2\left(\frac{xy}{ab}\right) cos\varphi = sin^2\varphi \qquad (5.3)$$

Special Cases:

<u>Case I.</u>

If φ = 0, 2π, 4π, …………..= 2nπ; where, n=0, 1, 2, 3, ………; (No phase difference)

Then $cos\varphi = 1$ and $sin\varphi = 0$. Now, from equation (5.3) we can write,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$$

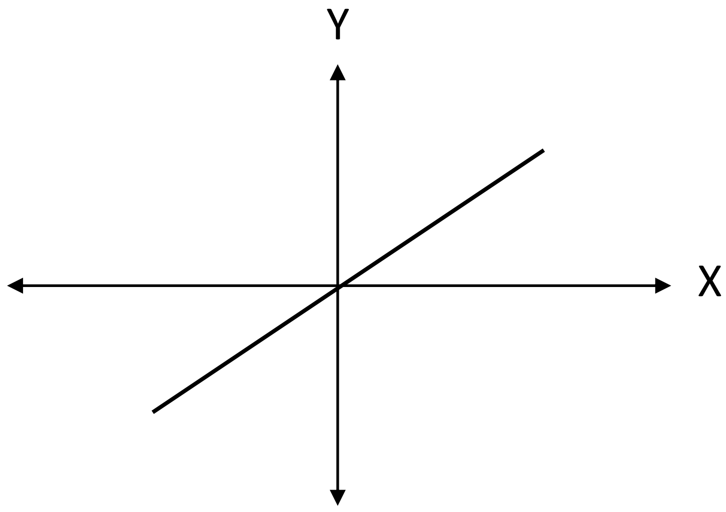
$$\Rightarrow \left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$$

$$\Rightarrow \pm\left(\frac{x}{a} - \frac{y}{b}\right) = 0$$

Thus, $y = \frac{b}{a}x \qquad (5.4)$

φ = 0

slope = $\frac{b}{a}$



Equation (5.4) represents a straight line passing through origin having a slope $\frac{b}{a}$.

Case II.

If φ = $\frac{\pi}{4}$ rad.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-2\left(\frac{xy}{ab}\right)cos\varphi = sin^2\varphi \qquad (5.3)$$

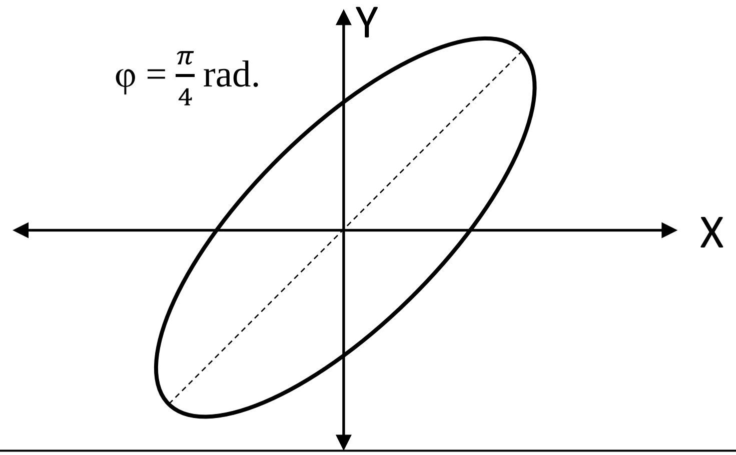
Then $cos\varphi = sin\varphi = \frac{1}{\sqrt{2}}$

Now, from equation (5.3) we can write,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{2}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{\sqrt{2}xy}{ab}=\frac{1}{2} \qquad (5.5)$$

Equation (5.5) represents an oblique ellipse whose length is parallel to X-axis is *2a* and breadth *2b.*



Case III.

If φ = $\frac{\pi}{2}$ rad.

Then $cos\varphi = 0$ *and* $sin\varphi = 1$

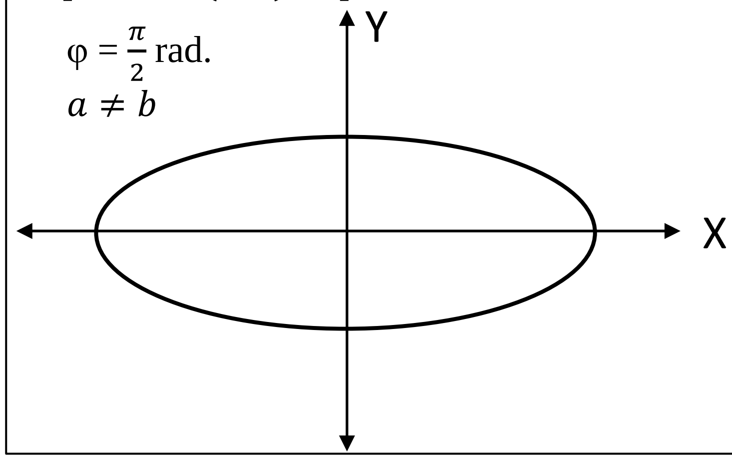
Now, from equation (5.3) we can write,

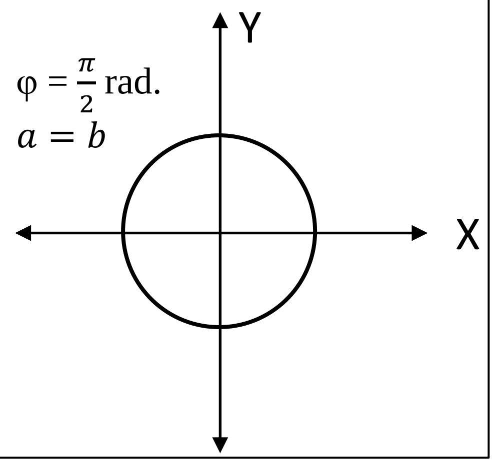
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (5.6)$$

Equation (5.6) represents a symmetric ellipse whose center coincide with the origin, length of the semi-major and semi-minor axes are *2a* and *2b*, respectively.

If $a = b, x^2 + y^2 = a^2$ (5.7)

Equation (5.7) represents a circle with radius a.





Case IV.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2\left(\frac{xy}{ab}\right) cos\varphi = sin^2\varphi \qquad (5.3)$$

If $\varphi = \frac{3\pi}{4}$ rad.

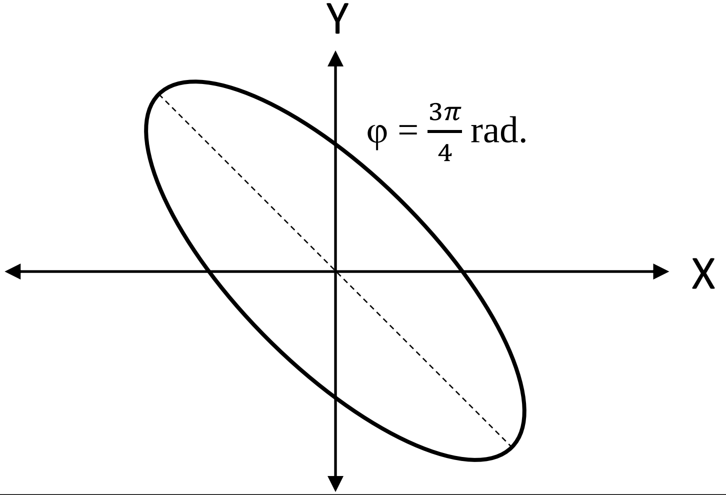
Then $cos\varphi$ = - $\frac{1}{\sqrt{2}}$ and $sin\varphi = \frac{1}{\sqrt{2}}$

Now, from equation (5.3) we can write,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{\sqrt{2}xy}{ab} = \frac{1}{2} \qquad (5.8)$$

Equation (5.8) represents an oblique ellipse.



Case V.

If $\varphi = \pi$ rad.

Then $cos\varphi$ = -1 and $sin\varphi$ = 0

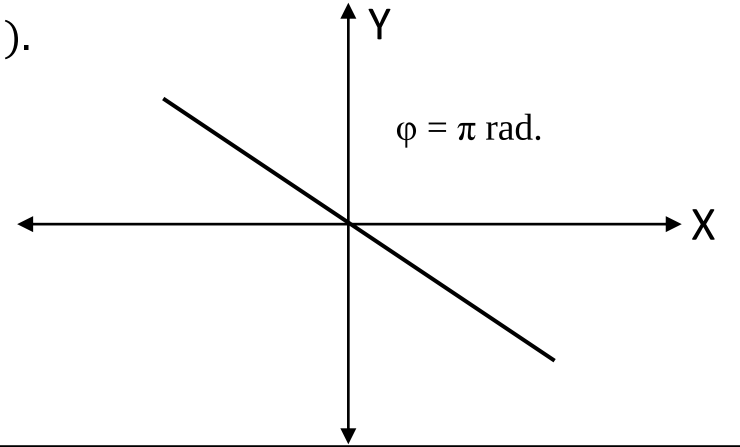
Now, from equation (5.3) we can write,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab} = 0$$

$$\Rightarrow \left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0 \Rightarrow \pm\left(\frac{x}{a} + \frac{y}{b}\right) = 0$$

Thus, $y = -\frac{b}{a}x$ (5.9)

Equation (5.9) represents a straight line passing through origin having a slope ($-\frac{b}{a}$).



Combination of two SHMs' at right angles to each other (frequency ratio 2:1, different phase and different amplitudes)

The equations of displacement of the SHMs' at right angles to each other having the frequency ratio 2:1 are as follows,

$x = a\ sin(2\omega t+\varphi)$ (5.10)

$y = b\ sin\omega t$ (5.11)

Let us relate the two equations of displacements as follows,

$$\frac{x}{a} = sin(2\omega t+\varphi) = sin2\omega t\ cos\varphi + cos2\omega t\ sin\varphi = 2sin\omega t\ cos\omega t\ cos\varphi + (1- 2sin^2\omega t)\ sin\varphi$$

$$= 2.\frac{y}{b}\sqrt{1-\frac{y^2}{b^2}}\ cos\varphi + \left(1-2.\frac{y^2}{b^2}\right) sin\varphi \quad \text{[from equation (5.11)]}$$

$$\left[\frac{x}{a} - \left(1-2.\frac{y^2}{b^2}\right) sin\varphi\right] = \frac{2y}{b} cos\varphi\sqrt{1-\frac{y^2}{b^2}}$$

Now, let us take square on both sides of the equation,

$$\left(\frac{x}{a} - sin\varphi\right)^2 + \frac{4y^4}{b^4} sin^2\varphi + 2\left(\frac{x}{a} - sin\varphi\right)\frac{2y^2}{b^2} sin\varphi = \frac{4y^2}{b^2}\left(1-\frac{y^2}{b^2}\right) cos^2\varphi$$

$$\Rightarrow \left(\frac{x}{a} - sin\varphi\right)^2 + \frac{4y^4}{b^4}(sin^2\varphi + cos^2\varphi) - \frac{4y^2}{b^2}(sin^2\varphi + cos^2\varphi) + \frac{4y^2}{b^2}\frac{x}{a} sin\varphi = 0$$

$$\Rightarrow \left(\frac{x}{a} - sin\varphi\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{x}{a} sin\varphi - 1\right) = 0 \quad (5.12)$$

Equation (5.12) is the general equation of a curve having two loops, for any phase and amplitude. The actual shape of curve will depend upon the phase difference φ.

$$\left(\frac{x}{a} - sin\varphi\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{x}{a}sin\varphi - 1\right) = 0 \qquad (5.12)$$

Case I.

φ = 0, π, 2π, etc.

sinφ = 0; Equation (5.12) becomes

$$\frac{x^2}{a^2} + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1\right) = 0 \qquad (5.13)$$

Equation (5.13) represents two loops like the shape of eight.

Case II.

$\varphi = \frac{\pi}{2}$

sinφ = 1; Equation (5.12) becomes

$$\left(\frac{x}{a} - 1\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{x}{a} - 1\right) = 0$$

$$\Rightarrow \left(\frac{x}{a} - 1\right)^2 + 2.\left(\frac{x}{a} - 1\right)\frac{2y^2}{b^2} + \frac{4y^4}{b^4} = 0$$

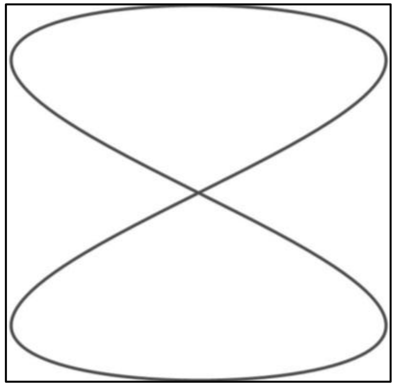
$$\Rightarrow \left[\left(\frac{x}{a} - 1\right) + \frac{2y^2}{b^2}\right]^2 = 0$$

$$\Rightarrow \left(\frac{x}{a} - 1\right) + \frac{2y^2}{b^2} = 0$$

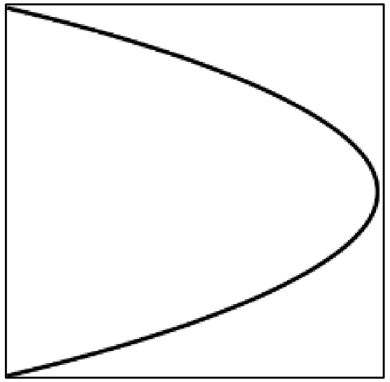
$$\Rightarrow \frac{2y^2}{b^2} = -\left(\frac{x}{a} - 1\right)$$

$$\Rightarrow y^2 = -\frac{b^2}{2a}(x - a) \qquad (5.14)$$

Equation (5.14) represents the equation of parabola with the vertex at (a, 0).



φ = 0



$\varphi = \frac{\pi}{2}$

$$\left(\frac{x}{a} - sin\varphi\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{x}{a}sin\varphi - 1\right) = 0 \qquad (5.12)$$

Case III.

$\varphi = \frac{3\pi}{2}$; $sin\varphi$ = -1; Equation (5.12) becomes

$$\left(\frac{x}{a} + 1\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - \frac{x}{a} - 1\right) = 0$$

$$\Rightarrow \left(\frac{x}{a} + 1\right)^2 - 2.\left(\frac{x}{a} + 1\right)\frac{2y^2}{b^2} + \frac{4y^4}{b^4} = 0$$

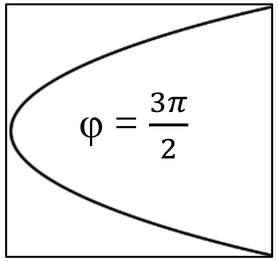
$$\Rightarrow \left[\left(\frac{x}{a} + 1\right) - \frac{2y^2}{b^2}\right]^2 = 0$$

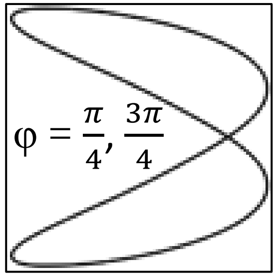
$$\Rightarrow \left(\frac{x}{a} + 1\right) + \frac{2y^2}{b^2} = 0$$

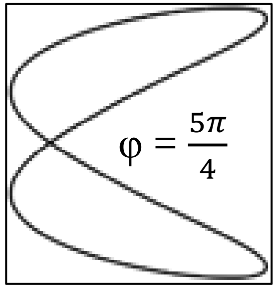
$$\Rightarrow \frac{2y^2}{b^2} = -\left(\frac{x}{a} + 1\right)$$

$$\Rightarrow y^2 = \frac{b^2}{2a}(x+a) \qquad (5.15)$$

Equation (5.15) represents the equation of parabola with the vertex at (-a, 0).







Case IV.

$\varphi = \frac{\pi}{4}$; $sin\varphi = \frac{1}{\sqrt{2}}$; Equation (5.12) becomes

$$\left(\frac{x}{a} - \frac{1}{\sqrt{2}}\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{x}{a}\frac{1}{\sqrt{2}} - 1\right) = 0$$

$$\Rightarrow \left(\frac{x}{a} - \frac{1}{\sqrt{2}}\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} + \frac{x}{\sqrt{2}a} - 1\right) = 0 \qquad (5.16)$$

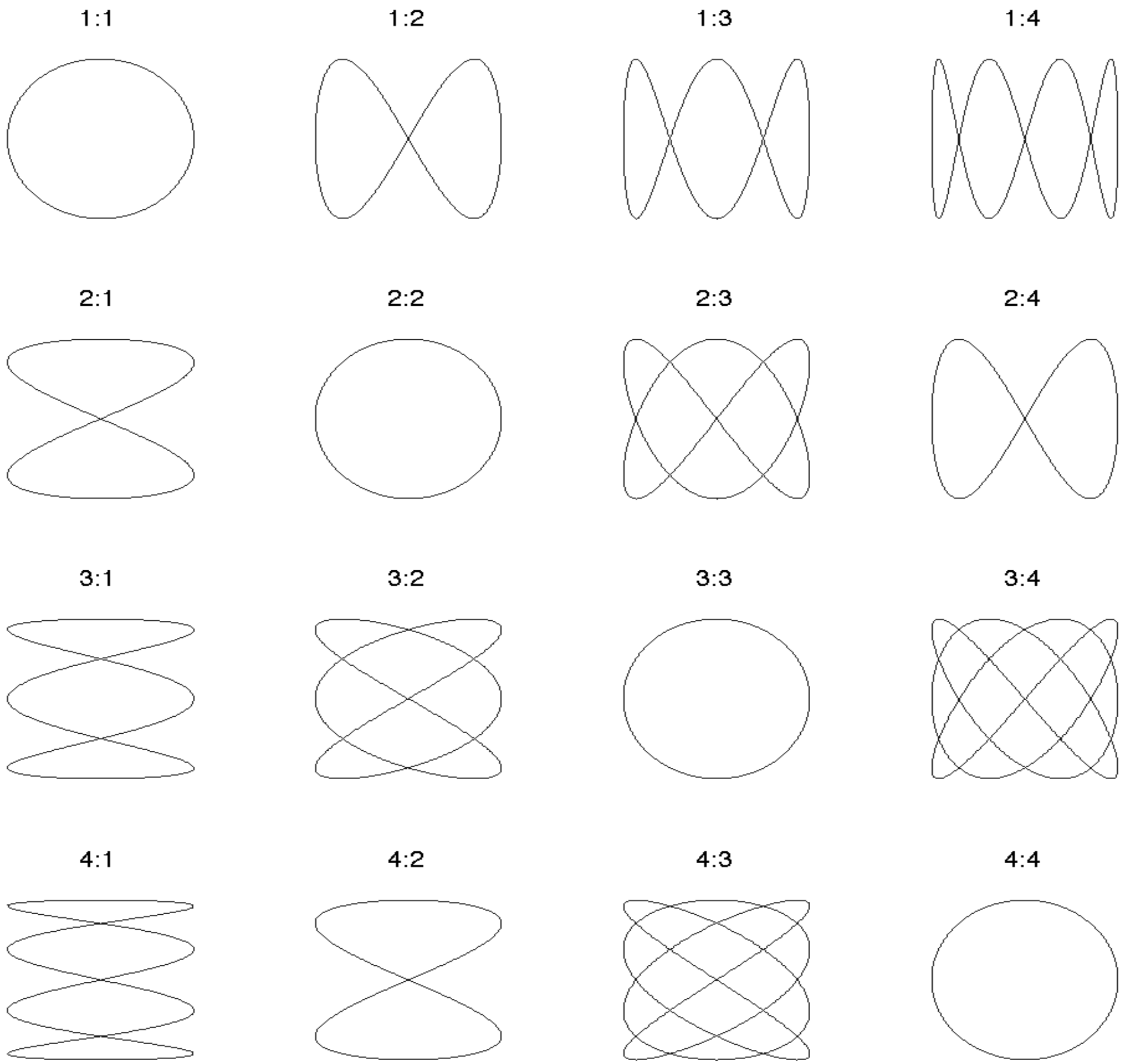
Equation (5.16) represents the equation of two equal loops skewed towards each other on the right side of the Y-axis.

Similarly, for $\varphi = \frac{3\pi}{4}$; $sin\varphi = \frac{1}{\sqrt{2}}$; results will be the same.

But for $\varphi = \frac{5\pi}{4}$; $sin\varphi = -\frac{1}{\sqrt{2}}$; Equation (5.12) becomes

$$\Rightarrow \left(\frac{x}{a} + \frac{1}{\sqrt{2}}\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - \frac{x}{\sqrt{2}a} - 1\right) = 0 \qquad (5.17)$$

Equation (5.17) represents the equation of two equal loops skewed towards each other on the left side of the Y-axis.



References:
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