Special Theory of Relativity

Introduction:

In 1905 Albert Einstein on his age at 25 showed how measurement of time and space are effected by motion between an observer and what is being observed. **Relativity connects space and time, matter and energy, electricity and magnetism.**
When we measure the length of a moving airplane from the ground we find it to be shorter than it is to somebody in the airplane itself.
The truth of the consequences of Newtonian mechanics is based on the approximation that the velocities of the particles of the system are small compared to the velocity of light. However, when the velocities involved approaches the speed of light the true description no longer represents the experimental facts and they must be altered to conform to what is called **special theory of relativity**.
Einstein showed that Maxwell's theory is consistent with special theory of relativity, whereas Newtonian mechanics is not, and his modification of mechanics brought these branches of physics into accord. As we will find, **relativistic and Newtonian mechanics agree for relative speed much lower than the speed of light.** For this Newtonian mechanics seemed correct for so long. **At higher speeds Newtonian mechanics fails and must be replaced by the relativistic version.**

Frame of reference:

Relative motion: Using a certain co-ordinate system in three dimensions, we characterize any location by three coordinate of that space point. When we say that something is moving, what we mean that it's position with respect to something is changing. That is, all motion is relative. In each case, a frame of reference is a part of the description of the system.

Examples: <u>Passengers move relative to an airplane; the airplane moves relative to the earth; the earth moves relative to the sun; the sun moves relative to the galaxy of stars and so on.</u>

A body in motion can be located with reference to some ordinate system called the frame of reference. If the coordinates of all the points of a body remain unchanged with time and with respect to frame of reference, the body is said to be at rest. On the other hand, if the coordinates of all the points of a body changed with time and with respect to frame of reference, the body is said to be in motion.

Inertial frame of reference:
An inertial frame of reference is one in which Newton's first law of motion holds. Any frame of reference that moves at constant velocity relative to and inertial frame is itself an inertial frame. It also called Galilean or Newtonian frame of reference.

Non-inertial Frame of Reference:
The frame relative to which an uncelebrated body appears accelerated is known as non-inertial frame. There is no universal frame of reference.

Special relativity deals with the inertial frame of reference, where the body moves with a constant velocity. Example: An airplane with constant velocity.

General relativity involves non-inertial frame of reference which is accelerated with respect to another. Example: Elevator, Merry go round.

Earth's velocity on its axis 30 m/s, around the sun 30 km/s.

Postulates of Special Theory of Relativity:

Two postulates underline special relativity. The first postulate is the Principle of relativity, and the second postulate is based on the results of many experiments.

1. Principle of equivalence of physical laws:

> ***"The laws of physics are the same in all inertial frames of reference"***

Explanation: This postulate follows form the absence of a universal frame of reference. One can only detect the relative motion of one material body with respect to the other and not absolute motion of bodies through the space. If the laws of physics were different for different observers in relative motion, the observers could not determine which of them were "stationary" in space and which were "moving".

2. Principle of constancy of velocity of light:

> ***"The speed of light in free space (vacuum) has the same value in all inertial frames of reference."***

It states that the speed of light is same in all directions and is independent of the source, intervening medium or the observed. That is regardless of their relative motion or of the motion of the source of the light *(No matter what the speed of the observer or the source)*. Thus a person moving toward or away from a source of light will measure the same speed for that light as someone at rest with respect the source.This postulate follows directly from the results of many experiments (such as Michelson-Morley experiment).

Galilean Transformation:

The transformation of co-ordinates of a particle from one inertial frame of reference to another is called the Galilean transformation.

Suppose we are in an inertial frame of reference S and the coordinates of some event that occurs at the time t are x, y, z. An observer located in a different inertial frame S′, which is moving with respect to S at constant velocity $\vec{u}$, will find that the same event occurs at the time t' and has the coordinates x', y', z'. For convenience, let $\vec{u}$ is in the $+x$ direction as shown in the figure.

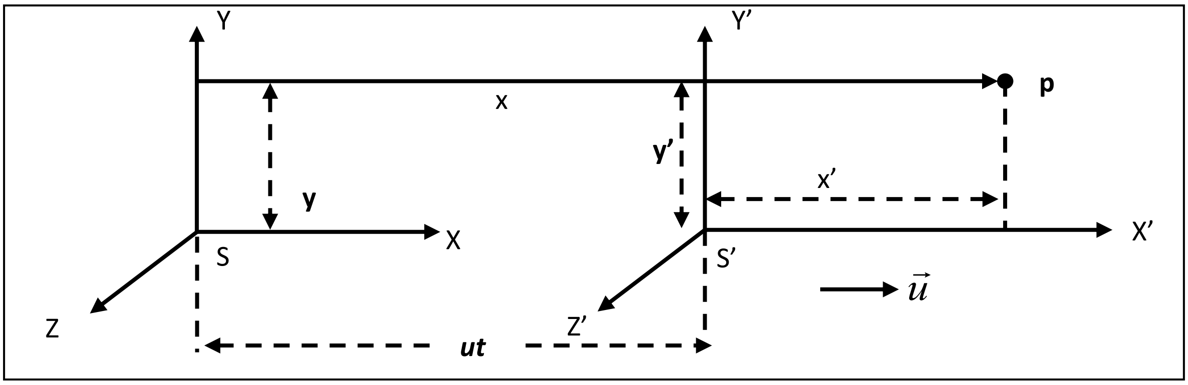


Fig.1: Frame S' moves in the +x direction with speed u relative to frame

If the clocks in both systems are started (t = 0) when the origins of S and S′ coincide, measurements in the x-direction made in S will be greater than those made in S′ by the amount of ut, which is the distance S′ has moved in the x-direction. That is,

$$x' = x - ut, \qquad \ldots.. \qquad \ldots\ldots \qquad \ldots.. \qquad [1a]$$

There is no relative motion in the y and z directions, and so

$$y' = y \quad ….. \quad …… \quad ….. \quad [1b]$$
$$z' = z \quad ….. \quad …… \quad ….. \quad [1c]$$

These set of equation is called the Galilean transformation.
It is assumed that time can be defined independently of any particular frame of reference. This is an implicit assumption of classical physics, which is expressed in the transformation equations by the absence of a transformation for t. We can make this assumption of the universal nature of time explicit by adding to the Galilean transformations the equation

$$t' = t \quad ….. \quad …… \quad ….. \quad [1d]$$

The set of equations [1] to [4] is known as Galilean transformation.
The transformation from S′ to S is

$$x = x' - ut', \quad ….. \quad …… \quad ….. \quad [2a]$$
$$y = y' \quad ….. \quad …… \quad ….. \quad [2b]$$
$$z = z' \quad ….. \quad …… \quad ….. \quad [2c]$$
$$t = t' \quad ….. \quad …… \quad ….. \quad [2d]$$

To convert velocity components measured in the S frame to their equivalents in the S' frame according to the Galilean transformation, we get

$$v_x' = \frac{dx'}{dt'} = \frac{d(x-ut)}{dt} = v_x - u \quad ….. \quad …… \quad ….. \quad [3a]$$
$$v_y' = \frac{dy'}{dt'} = \frac{dy}{dt} = v_y \quad ….…… \quad ….. \quad [3b]$$
$$v_z' = \frac{dz'}{dt'} = \frac{dz}{dt} = v_z \quad ….. …… \quad ….. \quad [3c]$$

We can see that different velocities are assigned to a particle by different observers when the observers in relative motion. **These are the velocity transformation equations correspond to Galilean transformation.**

The accelerations are

$a_x' = \frac{dv_x'}{dt'} = \frac{d(v_x - u)}{dt} = \frac{dv_x}{dt} - \frac{du}{dt} = a_x - 0 = a_x$, $a_y' = a_y$ and $a_z' = a_z$, that is the **transformation of acceleration is invariant under Galilean transformation**.

Similarly, the force acting on the particle of mass *m* observed by two inertial frames of reference S and S′ is invariant when u << c, given as $ma_x' = ma_x \Rightarrow F_x' = F_x$.

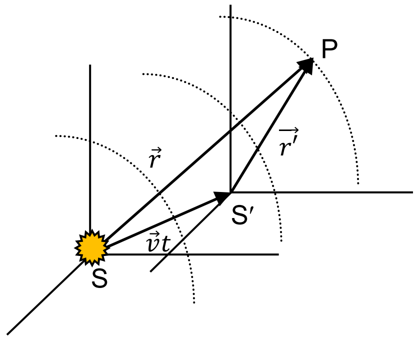
The second postulate calls for the same value of the speed of light c when determined in S or S′. If we measure the speed of light in the x-direction in the S system to be c, however, in the S′ system it will be

$$c' = c - u \qquad \text{according to Eq}^{n}\text{-[3a].}$$

Clearly a different transformation is required if the postulates of special relativity are to be satisfied.

Failure of Galilean Transformation

Although the Galilean transformation and the corresponding velocity transformation seem straightforward enough, they violate both the postulates of special relativity.



Let us consider a light source paced at the origin of unprimed system, is emitting spherical waves travelling with the speed of light c. Then the velocity of a point on a given wave surface in the unprimed system will be $\dot{\vec{r'}} = c\hat{n}$ .According to equation 3a, the corresponding wave velocity in the primed system is $\dot{\vec{r'}} = c\hat{n} - \vec{v}$. In this system the magnitude of the wave velocity with respect to the source of light in general will be c. Thus, it is clear that a different transformation is required if the postulates of special relativity are to be satisfied.

Thus the program of theory of relativity is therefore twofold. First, there must be obtained a transformation between two uniformly moving systems that will preserve the velocity of light, Second, the laws of Physics those do not keep their form invariant are to be generalized so as to obey the equivalence postulate.

LORENTZ TRANSFORMATION

What **Einstein**'s *special theory of relativity* says is that to understand why the *speed of light* is constant, we have to modify the way in which we translate the observation in one *inertial frame* to that of another. The *Galilean transformation*

$$x' = x - ut\,,\ t' = t \qquad [1]$$

is wrong. The correct relation is

$$x' = \frac{x-ut}{\sqrt{1-\frac{u^2}{c^2}}}\,, \qquad t' = \frac{t-\frac{u}{c^2}x}{\sqrt{1-\frac{u^2}{c^2}}} \qquad [2]$$

This is called the *Lorentz transformation*.

You can see that if the relative velocity u between the two frames is much smaller than the *speed of light c*, then the ratio *u/c* can be neglected in this relation and we recover the *Galilean transformation*. So the reason why we did not have any problems with the *Galilean transformation* up to now is that u was small enough for it to be a good approximation of the *Lorentz transformation*.

<u>Derivation of Lorentz Transformation Equations</u>

Let us consider a single event in frame S corresponds to a single event in frame S′. For this we can guess the nature of the correct relation between x and x′ is

$$x' = k(x - ut) \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad [3a]$$

Here *k* is a factor that does not depend upon either *x* or *t* but may be a function of *u*, *k* should be linear in x and x′, so an event in S corresponds to an event in S′. *k* should be simple and as so when v <<c, LT tends to GT.

Because the equations of physics must have the same form in both S and S′, in order to take into account the difference in the direction of relative motion we need only to change the sign of u to write the corresponding equation for x in terms of x′and t′:

$$x = k(x' + ut') \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad [4]$$

The factor k must be the same in both frames of references. As y, y′ and z, z′ are perpendicular to the direction of u, thus

$$y = y', \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad [3b]$$

$$z = z', \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad [3c]$$

Let us consider that at t = 0, S and S′ coincide, and a light pulse starts at the origin of Sand the observers in each system measure the speed with which the flare's light spreads out. When S and S′ coincide t′ = t = 0, and at this moment both observers must find the same speed c according to Einstein's postulates of relativity. This means that for the x-component of the wave front of the light pulse,

in frame S, $c = x/t$ i.e., $x = ct$ and
in frame S′, $c = x'/t'$ i.e., $x' = ct'$

Substituting ct for x and ct′ for x′ in Eqs. [3] and [4], we get

$$ct' = k(ct - ut) = k(c-u)t \quad \ldots \quad \ldots \quad [5]$$

and

$$ct = k(ct' + ut') = k(c+u)t' \quad \ldots \quad \ldots \quad [6]$$

From equation (5) we can write, $$\frac{t'}{t} = \frac{k(c-u)}{c} \quad \ldots \quad \ldots \quad [7]$$

Similarly, from equation (6), $$\frac{t'}{t} = \frac{c}{k(c+u)} \quad \ldots \quad \ldots \quad [8]$$

and thus we get,

$$\frac{k(c-u)}{c} = \frac{c}{k(c+u)} \Rightarrow k^2 = \frac{c^2}{c^2-u^2} \Rightarrow k^2 = \frac{1}{1-u^2/c^2} \Rightarrow k = \frac{1}{\sqrt{1-u^2/c^2}} \ldots\ldots \quad \ldots\ldots \quad [9]$$

Now from Equation 7 we can write, $$t' = \frac{kt(c-u)}{c} = \frac{tc-tu}{\left(\sqrt{1-\frac{u^2}{c^2}}\right)c} = \frac{t-\frac{tu}{c}}{\left(\sqrt{1-\frac{u^2}{c^2}}\right)} = \frac{t-\frac{ux}{c^2}}{\left(\sqrt{1-\frac{u^2}{c^2}}\right)} \ldots\ldots [10]$$

From [10], we can write $$t' = \frac{t - xu/c^2}{\sqrt{1-u^2/c^2}} = k(t - \frac{xu}{c^2}) \quad \ldots\ldots \quad \ldots \quad [11]$$

Or we can write $$t = \frac{t' + x'u/c^2}{\sqrt{1-u^2/c^2}} = k(t' + \frac{x'u}{c^2}) \quad \ldots\ldots \quad \ldots \quad [11]$$

The complete relativistic transformation is

$$x' = k(x - ut) \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad [12a]$$

$$y' = y \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad [12b]$$

$$z' = z \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad [12c]$$

$$t' = k(t - \frac{ux}{c^2}) \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad [12d]$$

Therefore, to obtain the inverse transformation, primed and unprimed quantities in Eqs. 12(a)-12(d) are exchanged and u is replaced by –u:

$$x = k(x' + ut'), \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad [13a]$$

$$y = y', \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad [13b]$$

$$z = z', \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad [13c]$$

$$t = k(t' + \frac{ux'}{c^2}), \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad [13d]$$

From Eq. [12d] & [13d], we can see that the time coordinate of one inertial system depends on both the time and space coordinates of another inertial system. Hence, instead of treating space and time separately, as is quite properly done in classical theory, it is natural in relativity to treat them together. Also we can see when u<<c, the Lorentz transformation corresponds to Galilean transformation.

Relativistic Velocity Transformation

The complete relativistic transformation is

$$x' = k(x - ut) \quad \ldots\ldots \quad [12a]$$
$$y' = y \quad \ldots\ldots \quad [12b]$$
$$z' = z \quad \ldots\ldots \quad [12c]$$
$$t' = k(t - \frac{ux}{c^2}) \quad \ldots\ldots \quad [12d]$$

By differentiating these inverse Lorentz transformation equations for x, y, z and t, we get

$$dx' = k(dx - udt),\ dy' = dy,\ dz' = dz,\ dt' = k(dt - u\frac{dx}{c^2}).$$

Therefore,

$$v'_x = \frac{dx'}{dt'} = \frac{k(dx - udt)}{k(dt - u\frac{dx}{c^2})} = \frac{\frac{dx}{dt} - u}{1 + \frac{u}{c^2}\frac{dx}{dt}} = \frac{v_x - u}{1 - \frac{uv_x}{c^2}} \quad \ldots\ \ldots\ldots\ [16a]$$

and, $$v'_y = \frac{dy'}{dt'} = \frac{dy}{k(dt - \frac{u}{c^2}dx)} = \frac{dy}{k(dt - \frac{u}{c^2}dx)} = \frac{\frac{dy}{dt}\sqrt{1 - u^2/c^2}}{1 - \frac{u}{c^2}\frac{dx}{dt}} = \frac{v_y\sqrt{1 - u^2/c^2}}{1 - \frac{uv_x}{c^2}} \quad \ldots\ldots\ [16b]$$

and, $$v_z' = \frac{dz'}{dt'} = \frac{dz}{k(dt - \frac{u}{c^2}dx)} = \frac{\frac{dz}{dt}\sqrt{1-u^2/c^2}}{1 - \frac{u}{c^2}\frac{dx}{dt}} = \frac{v_z\sqrt{1-u^2/c^2}}{1 - \frac{uv_x}{c^2}} \quad \ldots\ldots\ [16c]$$

If a ray of light is emitted in the moving frame S' in its direction of motion relative to S, then $v_x = c$, an observer in frame S will measure the speed

$$v'_x = \frac{v_x - u}{1 - \frac{uv_x}{c^2}} = \frac{c - u}{1 - \frac{uc}{c^2}} = c$$

Thus the observers in the car and on the road both find the same value for the speed of light, as they must.

We may write the complete set of relativistic velocity transformation equations corresponding to Lorentz transformation equations are

$$v'_x = \frac{v_x - u}{1 - \frac{u}{c^2}v_x}, \qquad v'_y = \frac{v_y\sqrt{1 - u^2/c^2}}{1 - \frac{u}{c^2}v_x}\ and, \qquad v'_z = \frac{v_z\sqrt{1 - u^2/c^2}}{1 - \frac{u}{c^2}v_x}$$

And the inverse velocity transformation equations are

$$v_x = \frac{v'_x + u}{1 + \frac{u}{c^2}v'_x},\quad v_y = \frac{v'_y\sqrt{1 - u^2/c^2}}{1 + \frac{u}{c^2}v'_x}\quad and,\ v_z = \frac{v'_z\sqrt{1 - u^2/c^2}}{1 + \frac{u}{c^2}v'_x}$$

Table: The relativistic velocity transformation equations

$v'_x = \dfrac{v_x - u}{1 - \dfrac{u}{c^2} v_x}$	$v_x = \dfrac{v'_x + u}{1 + \dfrac{u}{c^2} v'_x}$
$v'_y = \dfrac{v_y \sqrt{1 - u^2/c^2}}{1 - \dfrac{u}{c^2} v_x}$	$v_y = \dfrac{v'_y \sqrt{1 - u^2/c^2}}{1 + \dfrac{u}{c^2} v'_x}$
$v'_z = \dfrac{v_z \sqrt{1 - u^2/c^2}}{1 - \dfrac{u}{c^2} v_x}$	$v_z = \dfrac{v'_z \sqrt{1 - u^2/c^2}}{1 + \dfrac{u}{c^2} v'_x}$

Problems:

1. An observer S′ is moving to the right at a speed u = 0.50c away from the stationary observer S. The observer S′ measures velocity, v'_x, of a particle moving to the right away from her as shown in Fig. below. What speed, v, does the observer in S measure for the particle if a) v'_x= 0.60c? b) v'_x = 0.99c?

Solution:

The relative speed between the frames of references, u = 0.5 c

Speed of the particle in the S frame v_x =?

[a] The speed of the particle in the S′ frame v'_x = 0.6 c

We know, $v_x = \dfrac{v'_x + u}{1 + \dfrac{u}{c^2} v'_x}$

$$\therefore v_x = \frac{0.6c + 0.5c}{1 + \dfrac{0.5c}{c^2} 0.6} = \frac{1.1}{1.3} c = 0.846c$$

[b] The speed of the particle in S′ frame v'_x = 0.99 c

$$\therefore v_x = \frac{0.99c + 0.5c}{1 + \dfrac{0.5c}{c^2} 0.99c} = \frac{1.49}{1.495} c = 0.997c$$

Consequences of Special Relativity

There are four important consequences of Einstein's special relativity:

1. **Relativity of simultaneity**: Two events that appear simultaneous to an observer A will not be simultaneous to an observer B if B is moving with respect to A.
2. **Relativity of Time (Time dilation)**: Moving clocks tick slower than an observer's 'stationary' clock.
3. **Relativity of Length (Length contraction)**: Objects are observed to be shortened in the direction that they are moving with respect to the observer.
4. **Relativity of Mass and the Mass-energy equivalence**: The mass of a body moving at a speed u relative to a stationary observer is larger than its mass when at rest with respect to that observer. According to the relationship, $E = mc^2$, energy and mass are equivalent and transmutable.

Relativity of simultaneity

Two events that appear simultaneous to an observer A will not be simultaneous to an observer B if B is moving with respect to A.

Two events are said to be simultaneous if they occur at the same time. According to the relativity of simultaneity, if two observers are in relative motion, they will not agree as to whether two events are simultaneous. If one observer finds them to be simultaneous, the other generally will not, and conversely. The simultaneity of the two events is not an absolute concept and depends on the frame of reference. In fact, each observer is correct in his own frame of reference.

Let us consider an example to clarify the above statement:

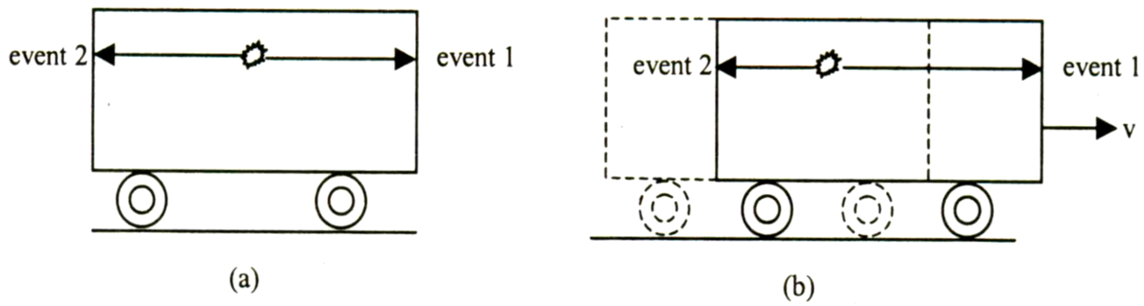


Figure: Spreading of light signals in a toy car as observed (a) by an observer on the car itself and (b) by an observer standing on the ground.

Imagine a trolley traveling at a constant speed along a smooth, straight track. In the center of the trolley there a light bulb is hanged. When it is switched on, the light spreads out in all directions at a speed c. Because the lamp is equidistant from the two ends, an observer on the trolley will find that the light reaches the front and the rear ends at the same time, i.e., the two events of light reaching the front and the rear ends occur simultaneously (Fig. a). However, to an observer on ground these two events do not appear to be simultaneous. As the light travels out from the bulb, the trolley itself moves forward, so the beam going to the rear end has a shorter distance to travel than the one going forward. According to this observer, therefore, the second event appears to happen before the first event (Fig. b). Therefore, we can conclude that the two events that are simultaneous in one inertial frame are not, in general, simultaneous in another frame.

In fact, measuring times and time intervals involve the concept of simultaneity and from the above discussion it follows that the time interval between two events may be different in different frames of references.

Proof of this statement:

Let us consider two frames of reference S and S′ . The frame reference S′ is moving with velocity u relative to the frame of reference S along +ve direction of x-axis and the two events occur simultaneously in S. Since the events are simultaneous in frame S, therefore we have $t_1 = t_2$. If t_1' and t_2' are the corresponding times of the same two events with respect to system S′, then we have from Lorentz transformation equations:- $t_1' = k(t_1 - \frac{ux_1}{c^2})$ and $t_2' = k(t_2 - \frac{ux_2}{c^2})$

$$\therefore \quad t_2' - t_1' = k(t_2 - \frac{ux_2}{c^2}) - k(t_1 - \frac{ux_1}{c^2}) = k\frac{u(x_1 - x_2)}{c^2} \quad \text{since } t_1 = t_2$$

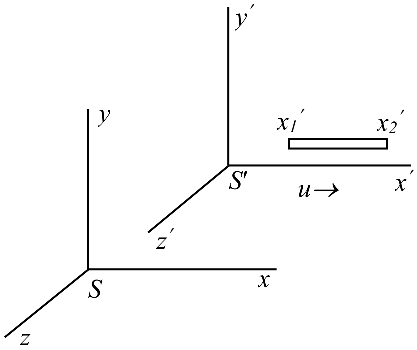
Thus, if the events are simultaneous in frame S′, t_1' must be equal to t_2' or $t_2' - t_1'$ must be equal to zero. But it is not so because x_1 is not equal x_2. Therefore, the same two events are not simultaneous in frame S′.

Lorentz-Fitzerald contraction (Length contraction)

Faster means shorter

Let us consider a rod lying along the x'axis in the moving frame S' which has a constant velocity u along the positive x direction only. An observer in this frame determines the coordinates of its ends to be x_1' and x_2', and so the proper length of the rod is

$L_0' = x_2' - x_1'$.



In order to find $L = x_2 - x_1$, the length of the rod as measured in the stationary frame S at time t, we can use the Lorentz transformation equation Eq.[13a]

$x_1' = k(x_1 - ut)$, and $x_2' = k(x_2 - ut)$.

Hence,

$$L = x_2 - x_1 = \frac{x_2'}{k} + ut - \frac{x_1'}{k} - ut = \frac{x_2' - x_1'}{k} = \frac{L_0}{k} = L_0\sqrt{1 - u^2/c^2}$$

$$\therefore \quad L = L_0\sqrt{1 - u^2/c^2} \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots [15]$$

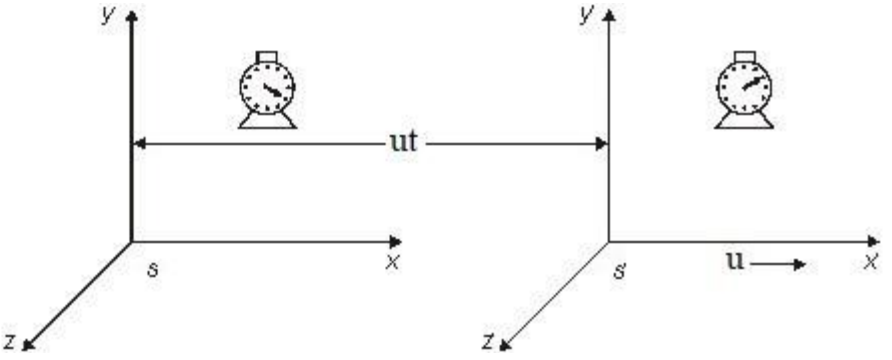
Eqn. [15] summarizes the effect known as length contraction. Observer S' who is at rest with respect to the object, measures the rest length L_0 (also known as **proper length**, in analogy with the proper time). All observes in motion relative to S' measure a shorter length, but only along the direction of motion; length measurements transverse to the direction of the motion are unaffected.

For ordinary speeds (u<<c), the effects of length contraction are too small to be compared.

1. An astronaut whose height on the earth is exactly 6 ft is lying parallel to the axis of a spacecraft moving at 0.90c relative to the earth. What is his height as measured by an observer in the same spacecraft? By an observer on the earth?
2. A spacecraft antenna is at an angle of 10° relative to the axis of the spacecraft. If the spacecraft moves away from the earth at a speed of 0.70c, what is the angle of the antenna as seen from the earth?

Relativity of Time (Time Dilation):

Moving clocks tick slower than an observer's "stationary" clock.



Let us consider a clock at the point x' in the moving frame S'. When an observer in S' finds that the time is t_1' an observer in S will find it to be t_1, where from Eq.[12d]

$$t_1 = k(t_1' + ux'/c^2)$$

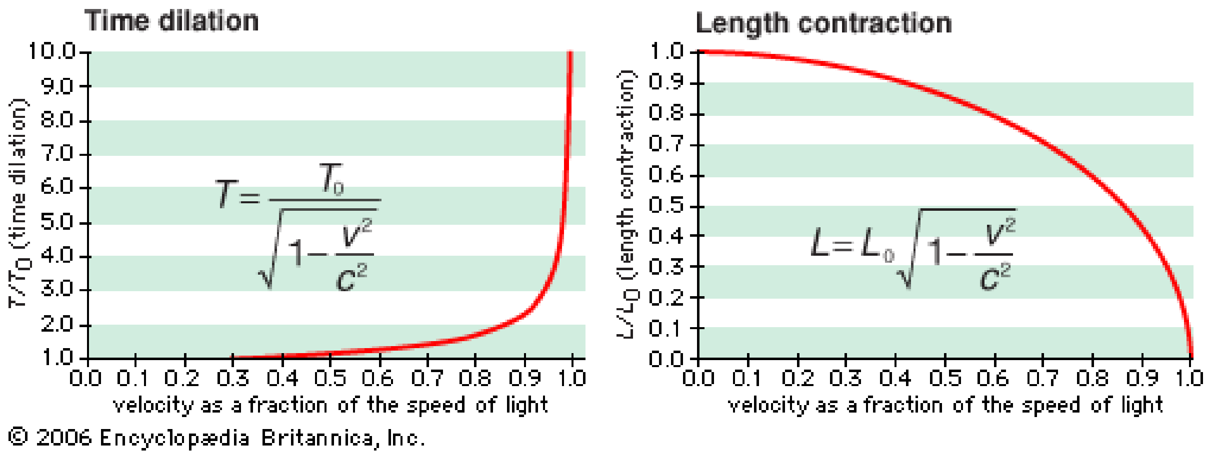
After a time, interval of Δt_0 (to observer in S'), the observer in the moving system S' finds that the time is now t_2' according to his clock. That is, $\Delta t_0 = t_2' - t_1'$.
The observer in S, however, measures the same time to be

$$t_2 = k(t_2' + ux'/c^2).$$

So, to observer in S the duration of time interval t is

$$\Delta t = t_2 - t_1 = k(t_2' - t_1') = kt_0 = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}} \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots [14]$$

Eq. [14] summarizes the effect known as **time dilation**. According to Eq. [14], observer in S measures a longer time interval than the observer in S' measures. This is a general result within special relativity, which we can summarize as follows:
Consider an occurrence that has a duration Δt_o. An observer S' fixed with respect to that occurrence (i.e. its beginning and end take place at the same point in space, according to S') measures the interval **Δt_0, which is known as the proper time**. An observer S moving with respect to S' measures a longer time interval Δt for the same occurrence. The interval Δt is always longer than Δt_0, no matter what the magnitude or direction of u is.
The time dilation effect has been verified experimentally with decaying elementary particles as well as with precise atomic clocks carried aboard aircraft.



3. How much time does a meter stick moving at 0.100c relative to an observer take to pass the observer? The meter stick is parallel to its direction of motion.
4. A woman leaves the earth in a spacecraft that makes a round trip to the nearest star, 4 light year distant, at a speed of 0.9c. How much younger is she upon her return than her twin sister who remained behind?

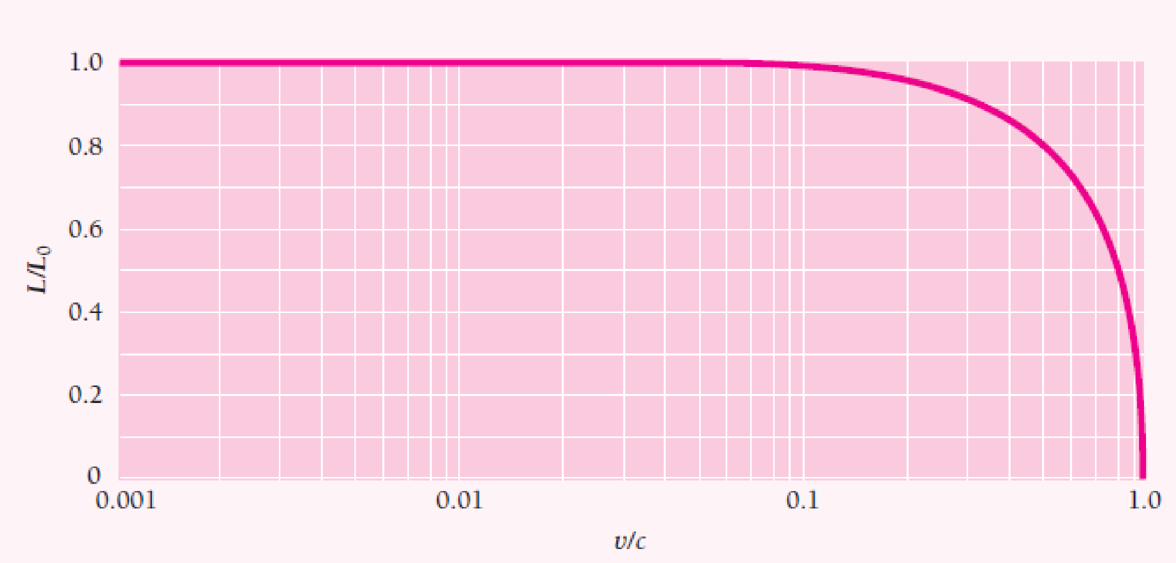


Figure 1.10 Relativistic length contraction. Only lengths in the direction of motion are affected. The horizontal scale is logarithmic.

Relativity of Mass

To investigate what happens to the mass of an object as its speed increases, let us consider an elastic collision (that is a collision in which kinetic energy is conserved) between two particles A and B as witnessed by observers in the reference frames S and S′ as shown in figure below, which are in uniform relative motion .

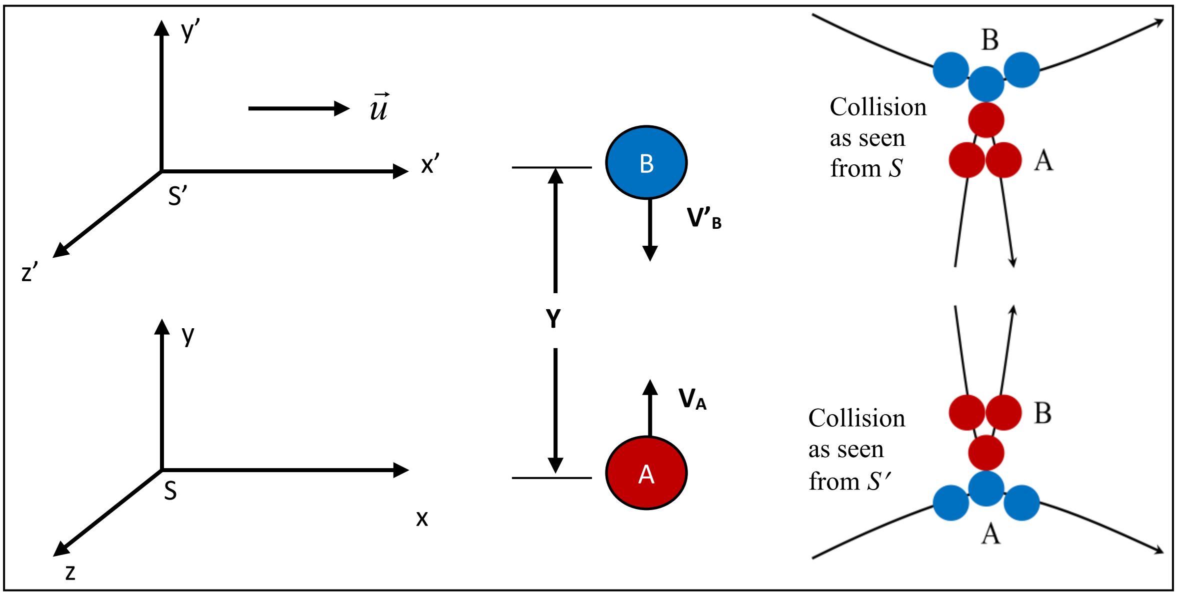


Figure: An elastic collision as observed in two different frames of reference. The balls are initially Y apart, which is the same distance in both frames since S' moves only in the X direction.

The properties of A and B are identical when determined in reference frames in which they are at rest. The frame S′ is moving in the +x direction with respect to S at the velocity *u*.

Before the collision, particle A had been at rest in frame S and particle B in frame S′. Then, at the same instant, A was thrown in the +y direction at the speed V_A while B was thrown in the -Y direction at the speed $\vec{V}_B$, where $V_A = V'_B$ .... [21]
Hence the behavior of A as seen from S is exactly the same as the behavior of B as seen from S′.

When the two particles collide, A rebounds in the -Y direction at the speed V_A while B rebounds in the +Y direction at the speed V_B . If the particles are thrown from positions Y apart, an observer in S finds that the collision occurs at y=Y/2 and one in S′ finds that it occurs at y′ = y = Y/2. The round-trip time t_0 for A as measured in S is therefore,

$$t_0 = \frac{Y}{V_A} \quad .... \quad .... \quad .... \quad [22]$$

and it is the same for B in S′: $t_0 = \dfrac{Y}{V'_B}$ .... [23]

If the linear momentum is conserved in the S frame, it must be true that

$$m_A V_A = m_B V_B \quad .... \quad .... \quad .... \quad [24]$$

where m_A and m_B are the masses of A and B and V_A and V_B are their velocities as measured in the S frame. In S, speed V_B is found from

$$V_B = \frac{Y}{t} \quad .... \quad .... \quad .... \quad [25]$$

where t is the time required for B to make its round trip as measured in S. In S′, however, the trip of B requires the time t_0 , where

$$t = \frac{t_0}{\sqrt{1-u^2/c^2}} \quad .... \quad .... \quad .... \quad [26]$$

From [25], $V_B = \dfrac{Y\sqrt{1-u^2/c^2}}{t_0} = V_A\sqrt{1-u^2/c^2}$ .... [27] since from [22] we know that $V_A = \dfrac{Y}{t_0}$.

Inserting these expressions for V_A and V_B in [24], we see that the momentum is conserved provided that

$$m_A = m_B\sqrt{1-u^2/c^2} \quad .... \quad .... \quad .... \quad [28]$$

Because A and B are identical when at rest with respect to an observer, the difference between m_A and m_B means that measurements of mass like those of space and time depend upon the relative speed between an observer and whatever he or she is observing.

In this example, both A and B are moving in S. In order to obtain a formula that gives the mass m of a body measured while in motion in terms of its mass m_o when measured at rest, we need only consider a similar example in which V_A and V_B are very small compared with v. In this case an observer in S will see B approach A with the velocity $\vec{v}$ and making a glancing collision (since $V_B << v$) and then continue on.

In S $m_A = m_0$ and $m_B = m$

and so Relativistic mass $m = \dfrac{m_0}{\sqrt{1-u^2/c^2}}$ .... [29]

The mass of a body moving at the speed v relative to an observer is larger than its mass when at rest relative to the observer by the factor

$$1\Big/\sqrt{1-u^2/c^2}\ .$$

This mass increase is reciprocal, to an observer in S', m_A= m and m_B=m_0. Measured from the earth, a spacecraft in flight is shorter than its twin still on the ground and its mass is greater. To somebody on the spacecraft in flight, the ship on the ground also appears to be shorter and to have a greater mass.

Relativistic mass increases are significant only at speeds approaching that of light. At a speed one tenth that of light the mass increase amounts to 0.5 percent but this increase is over 100% at a speed nine-tenths that of light. Only atomic particles such as electrons, protons, mesons and so on have sufficiently highly speeds for relativistic effects to be measurable and in dealing with these particles, the ordinary laws of physics cannot be used.

EQUIVALENCE OF MASS AND ENERGY:

Derive an equation for the relativistic kinetic energy of a body. How does this equation lead to the Einstein mass-energy equivalence $E = mc^2$?
Explain the significance of the following conditions:

(i) *show that the kinetic energy is $K \approx \frac{1}{2}m_0v^2$ for low speeds; for $v/c \ll 1$,*

(ii) *A mass less particle with a speed less than that of light can have neither energy nor momentum;*

(iii) *E and p can have any values for a mass-less particles like photon if the particle is moving a speed of light (u = c) and*

(iv) *the relationship between the energy and momentum of a particle is $E = \sqrt{m_0{}^2c^4 + p^2c^2}$*

Let a force F is acting on a body so that there is a displacement ds along the direction of the force. The work done is then given by $W = Fds$

If no other forces act on the object and the object starts from rest, all the work done on it becomes kinetic energy K. and is given by $K = \int_0^s Fds$ (1)

If the force is changing with time and is given by $F = \frac{dp}{dt} = \frac{d(mv)}{dt}$. Where v is the velocity of the body and the relativistic momentum is $p = mv = \frac{m_0v}{\sqrt{1-v^2/c^2}}$. So the kinetic energy K becomes

$$K = \int_0^S Fds = \int_0^S \frac{d(mv)}{dt}ds = \int_0^v vd[mv] = \int_0^v vd\left(\frac{m_0v}{\sqrt{1-v^2/c^2}}\right) \tag{2}$$

Integrating by parts $\int xdy = xy - \int ydx$, we get

Product Rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$\int \frac{d}{dx}(uv)dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$

$uv = \int udv + \int vdu$

$\int udv = uv - \int vdu$

$$K = \left[\frac{m_0vv}{\sqrt{1-v^2/c^2}}\right]_0^v - \int_0^v \frac{m_0vdv}{\sqrt{1-v^2/c^2}}$$

$$= \frac{m_0v^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0\int_0^v \frac{vdv}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_0v^2}{\sqrt{1-\frac{v^2}{c^2}}} + \left[m_0c^2\sqrt{1-\frac{v^2}{c^2}}\right]_0^v$$

$$= \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2 = mc^2 - m_0c^2 \tag{3}$$

This is the relativistic equation for kinetic energy.

The above equation may be written as

Total Energy, $mc^2 = m_0c^2 + K$ (4)

Let mc^2 is the total energy E of the object. We see that when the body is in at rest, then $K = 0$ and the body has an amount of

Rest energy, $E_0 = m_0c^2$ (5)

Thus, we can write $E = E_0 + K$ (6)

If the body is moving, its total energy is $E = mc^2 = \frac{m_0c^2}{\sqrt{1-v^2/c^2}}$ (7)

This is the Einstein famous mass-energy relation.

$\int u(x)v(x)\,dx = u(x)\int v(x)\,dx - \int\left[\, u'(x)\int v(x)\,dx\,\right]dx$

(i) For low speeds, $v/c << 1$, Then we can write

$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2 = m_0c^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right) \tag{8}$$

Using the binomial approximation $(1+x)^n \approx 1+nx$ with $|x| << 1$, we get

$$\frac{1}{\sqrt{1-v^2/c^2}} = \left(1-v^2/c^2\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \qquad \text{with} \qquad v/c << 1. \tag{9}$$

Thus the Kinetic energy is then

$$K = m_0c^2\left(\cancel{1} + \frac{1}{2}\frac{v^2}{c^2} - \cancel{1}\right) \approx \frac{1}{2}m_0v^2 \qquad \text{with} \qquad v/c << 1. \tag{10}$$

(ii) Mass-less particles:

We know total energy, $E = \dfrac{m_0c^2}{\sqrt{1-v^2/c^2}}$, (11)

andthe relativistic momentum , $p = \dfrac{m_0u}{\sqrt{1-v^2/c^2}}$ (12)

Case 1: When $m_0 = 0$ and v<c, i.e. E = p = 0 i.e.,**A mass less particle with a speed less than that of light can have neither energy nor momentum.**

Case 2: When $m_o = 0$ and v =c, E=0/0 and p=0/0 which are indeterminate: E and p can have any values. The equations [11] & [12] are consistent with the existence of mass-less particles that possess energy and momentum provided that they travel with the speed of light.

(iii) **Momentum of a mass-less particle**: There is another restriction on mass-less particles.

We can write $$E^2 = \frac{m_0{}^2c^4}{1-v^2/c^2}$$

Also we can write $p^2 = \dfrac{m_0{}^2v^2}{1-v^2/c^2}$, or, $p^2c^2 = \dfrac{m_0{}^2v^2c^2}{1-v^2/c^2}$

Subtracting p^2c^2 from E^2 yields

$$\begin{aligned} E^2 - p^2c^2 &= \frac{m_0{}^2c^4}{1-v^2/c^2} - \frac{m_0{}^2v^2c^2}{1-v^2/c^2} \\ &= m_0{}^2c^2\left(\frac{c^2}{c^2-v^2} - \frac{v^2}{c^2-v^2}\right)c^2 \\ &= m_0{}^2c^4\left(\frac{c^2-v^2}{c^2-v^2}\right) \\ &= m_0{}^2c^4 \end{aligned} \tag{13}$$

$$\therefore \qquad E^2 = m_0{}^2c^4 + p^2c^2$$

Therefore, for all particles, we have the **relativistic energy**

$$E = \sqrt{m_o{}^2c^4 + p^2c^2} = \sqrt{E_o{}^2 + p^2c^2} \tag{14}$$

According to this formula, if a particle exists with $m_0 = 0$, the relationship between its energy and momentum must be given by

$$E = pc \tag{15}$$

$\therefore$Momentum of a mass-less particle is, $p = E/c$ (16)

In fact, mass-less particles of two different kinds - the photon and the neutrino have indeed been discovered and their behavior is as expected.

1. **Find the mass of an electron (m_o=9.1x10^{-31} kg) whose velocity is 0.99c.**
   Solution:
   Given u/c=0.99, so $m = \dfrac{m_0}{\sqrt{1-u^2/c^2}} = \dfrac{9.1x10^{-31}}{\sqrt{1-(0.99)^2}} = 64x10^{-31}kg.$
   which is 7 times greater than the electron's rest mass. If v =c, m = ∞, from which we can conclude that v can never equal c:

   No material object can travel as fast as light

2. **A stationary body explodes into two fragments each of rest mass 1.0 Kg that move apart at speeds 0.6C relative to the original body. Find the rest mass of the original body.**
   Solution:
   The total energy of the original body must equal the sum of the total energies of the fragments. Hence
   $E = E_0 + K.E = E_0 + 0 = E_0$
   $$m_0c^2 = \frac{m_{01}c^2}{\sqrt{1-u_1^{\ 2}/c^2}} + \frac{m_{02}c^2}{\sqrt{1-u_2^{\ 2}/c^2}} = \frac{(1.0kg)c^2}{\sqrt{1-(0.6)^2}} + \frac{(1.0kg)c^2}{\sqrt{1-(0.6)^2}}$$
   Therefore,
   $m_0 = 2.5kg$
   Mass can be created or destroyed but when this happens an equivalent amount of energy simultaneously vanishes or comes into being and vice versa.

   Mass and energy are different aspects of the same thing

   The conversion factor between the unit of mass (the kg) and the unit of energy (the joule, J) is c^2. So, 1 kg of matter has an energy content of
   $m_0c^2 = (1kg)*(3x10^8 m/s)^2 = 9x10^{16} J$
   This is enough to send a payload of a million tons to the moon.

3. **An electron (m_0 = 0.511 MeV/c^2) and a photon (m_0 =0) both have momenta of 2.00 MeV/c. Find the total energy of each.**
   Solution:
   The electron's total energy is
   $E = \sqrt{m_0^2c^4 + p^2c^2} = \sqrt{(0.511MeV/c^2)^2c^4 + (2.00MeV/c)^2c^2} = 2.064\ MeV$

4. **What is the percentage increase in the mass of an electron accelerated to a K.E of 500 MeV? Use rest mass of electron = 0.511 MeV/c^2.**

5. **What is the speed of a particle if its kinetic energy is 1% larger than $\frac{1}{2}m_ou^2$?**

Doppler Effect:

The normal Doppler shift for waves such as sound which move with velocities v much less than c is given by the expression

Doppler effect for sound

$$f_{observed} = \left[\frac{v}{v \pm v_{source}}\right] f_{source}$$

which when changed to the normal notation for light waves becomes

$$\nu_{observed} = \nu_{source}\left[\frac{1}{1 \pm \frac{v_{source}}{v_{wave}}}\right]$$

where the plus sign is taken for waves traveling away from the observer. For light and other electromagnetic waves, the relationship must be modified to be consistent with the Lorentz transformation and the expression becomes

Doppler effect for light

$$\nu_{observed} = \left[\frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}}\right]\nu_{source}$$

which can be rearranged to the form

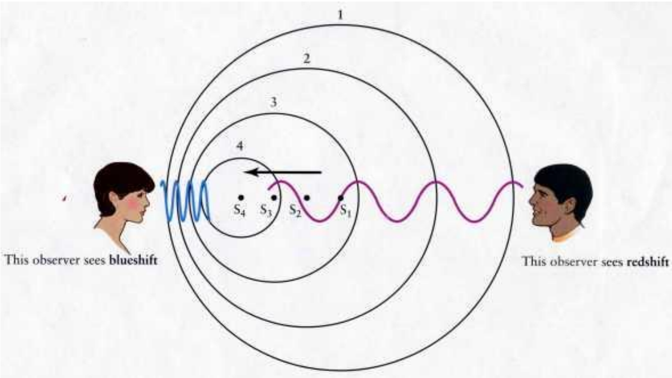
$$\nu_{observed} = \nu_{source}\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

or in common relativity notation: $\nu_{observed} = \nu_{source}\sqrt{\frac{1+\beta}{1-\beta}}$

Here v is the relative velocity of source and observer and v is considered positive when the source is approaching.

1. θ=0° : $\nu = \nu'\sqrt{\frac{1+\beta}{1-\beta}}$
2. θ=180° : $\nu = \nu'\sqrt{\frac{1-\beta}{1+\beta}}$
3. θ=90° : $\boxed{\nu = \nu'\sqrt{1-\beta^2}}$

Transverse doppler effect



Problem 1: A bus is moving towards john at 1.5×10^8 m/s.The bus flashes the headlight on john of frequency 4200 Hz. At what frequency will John receive the light?

Solution:
7800 Hz

Problem2: A galaxy in the constellation Ursa Major is receding from the earth at 15,000 km/s. If one of the characteristic wavelengths of the light the galaxy emits is 550 nm, what is the corresponding wavelength measured by astronomers on the earth?

Michelson-Morley Experiment

Ether: According to Maxwell, light waves are transverse electromagnetic wave. A material medium is necessary for the propagation of transverse waves. For light, it was assumed 'limuniferous ether'.

Tranverse waves require shearing and these forces can occur in solid only. It means that ether must be rigid solid filling the whole space. As the velocity of wave propagation depends on the elasticity of the medium, ether must be highly elastic. Thus, the whole free space must be filled up with this medium called ether which is difficult to conceive.

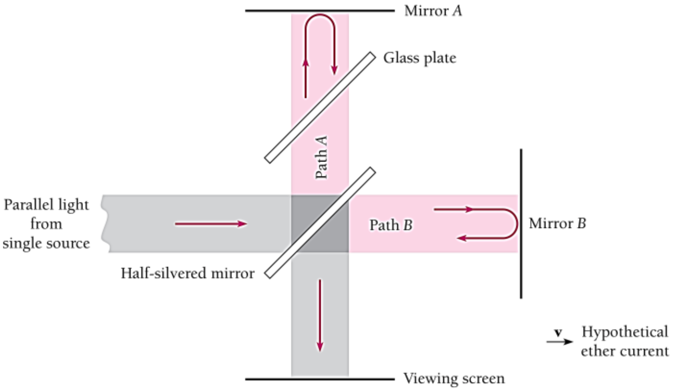
If light is propagated through the ether medium, thenin optical experiments, one would expect change in velocity (drift) depends upon the direction of light propagation with respect to motion of the apparatus.

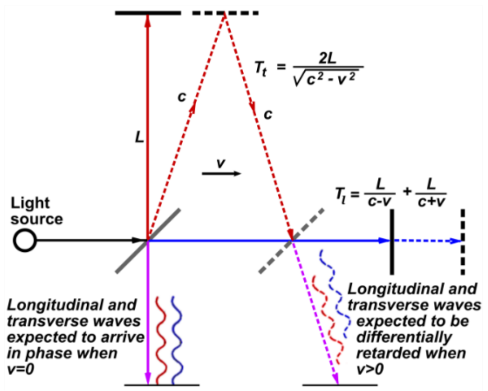
Michelson's most significant achievement, carried out in 1887 in collaboration with Edward Morley, was an experiment to measure the motion of the earth through the 'ether', a hypothetical medium pervading the whole universe in which light waves were supposed to occur. The notion of the ether was a hangover from the days before light waves recognized as electromagnetic, but nobody at the time seemed willing to discard the idea that light propagates relative to some sort of universal frame of reference.

The experiment:

To look for the earth's motion through the ether, Michelson and Morley used a pair of light beams formed by half-silvered mirror as shown in figure.

One light is directed to a mirror along a path perpendicular to the ether current, and the other goes to a mirror along a path parallel to the ether current. Both beams end up at the same viewing screen. The clear glass plates ensure that both beam pass through the same thickness of air and glass.If the transit times of the two beams are the same, they will arrive at the screen in phase and will interfere constructively. Ether current due to the earth's motion parallel to one of the beams, however, would cause the beams to have different transit times and the result would be destructive interference at the screen. This is the essence of the experiment.



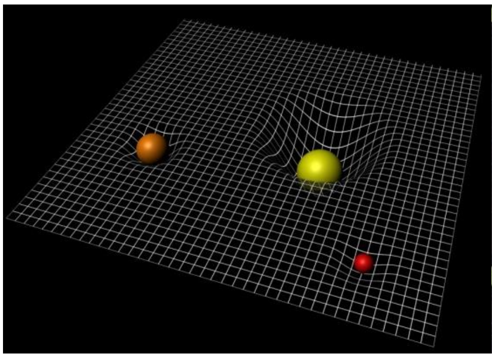


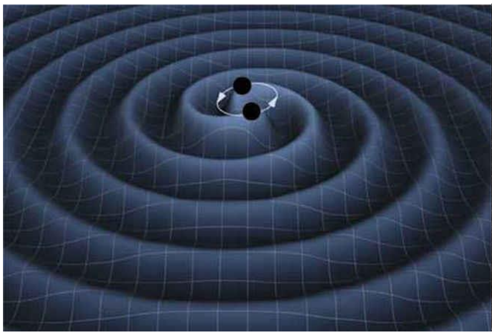
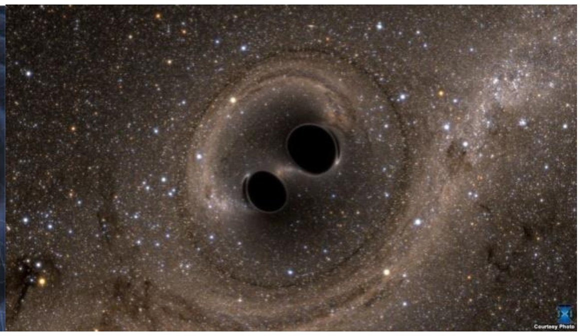
Although the experiment was sensitive enough to detect the expected ether drift, to everyone's surprise none was found. The negative result had two consequences.

- First, it showed that the ether does not exist and so there is no such thing as "absolute motion" relative to ether: all motion is relative to a specified frame of reference, not to a universal one.

- Second, the result showed that the speed of light is the same for all observers, which is not true for waves like sound and water waves that need a material medium in which to occur.

Space-Time and Detection of Gravitational Waves



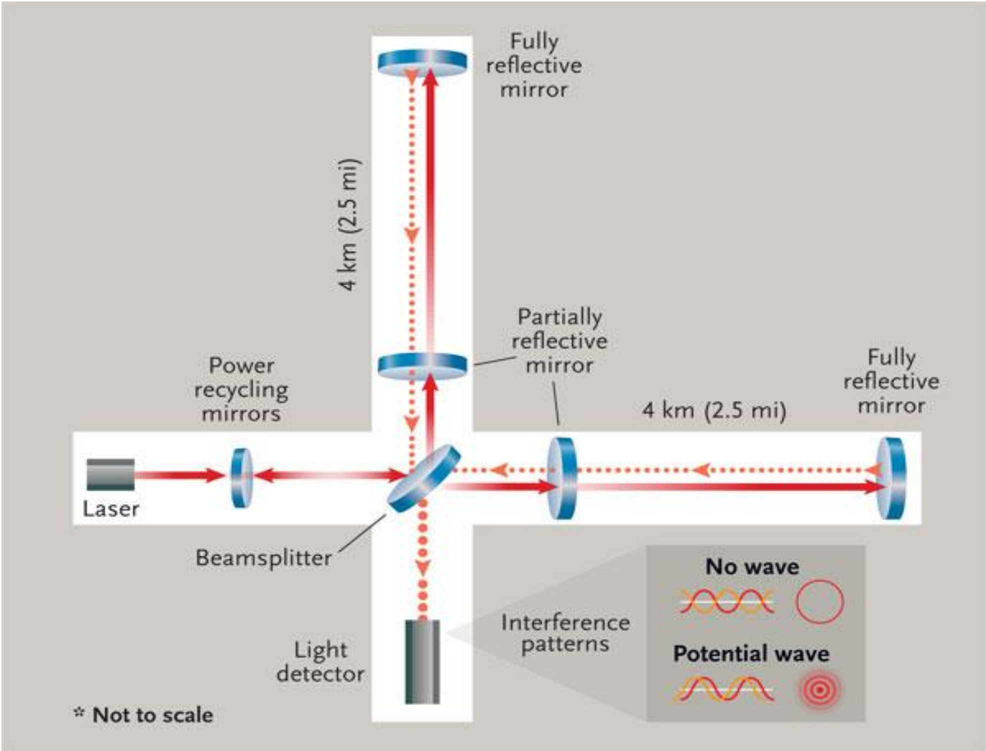




Gravitational Wave Detection by LIGO
(Laser Interferometer Gravitational-Wave Observatory)

Recently, physicists announced the first-ever direct detection of gravitational waves, ripples in the fabric of space-time predicted by Einstein's general theory of relativity. Two massive accelerating objects — in this case, a pair of stellar-mass black holes in a death-spiral — passed through space-time like paddles sweeping through water, creating vibrations that could (barely) be felt on Earth. The results are published in *Physical Review Letters*.

LIGO scientists have announced the direct detection of gravitational waves, a discovery that won't just open a new window on the cosmos — it'll smash the door wide open.

LIGO's Interferometer



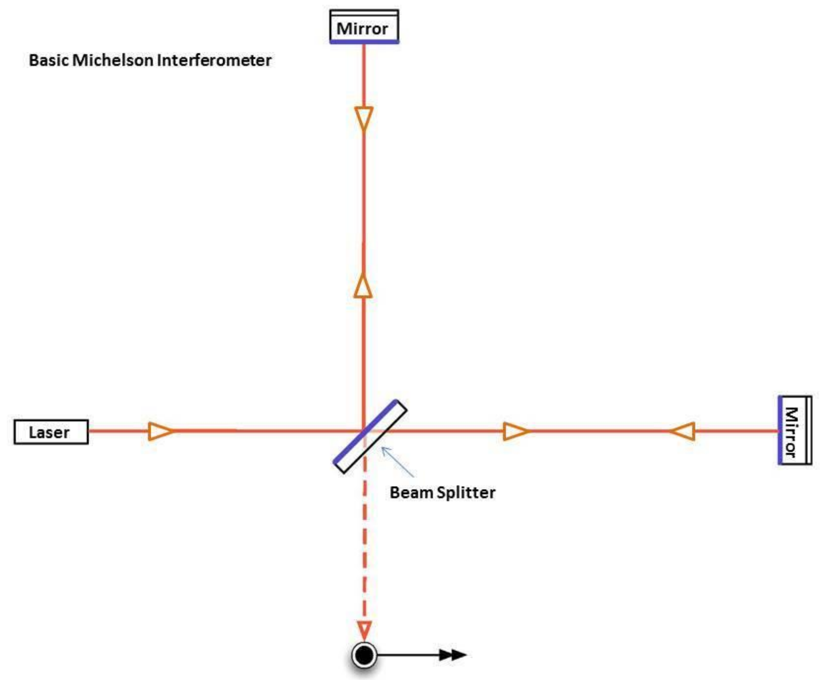
- LIGO consists of two L-shaped facilities, one near Hanford, Washington, and the other near Livingston, Louisiana. At 5:51 a.m. (EDT) on September 14, 2015, both labs caught the gravitational-wave signature of two colliding black holes, shortly after both facilities were turned on following five years of intensive upgrades.
- In this schematic of LIGO, a beamsplitter sends light along two paths perpendicular to each other. Each beam bounces between two mirrors, one of which allows a fraction of the light through. When the two transmitted beams meet and interfere, they'll cancel each other out — if the length of the path they've each traveled has remained constant. But if a gravitational wave passes through, it'll warp space-time and change that distance, creating an interference pattern.

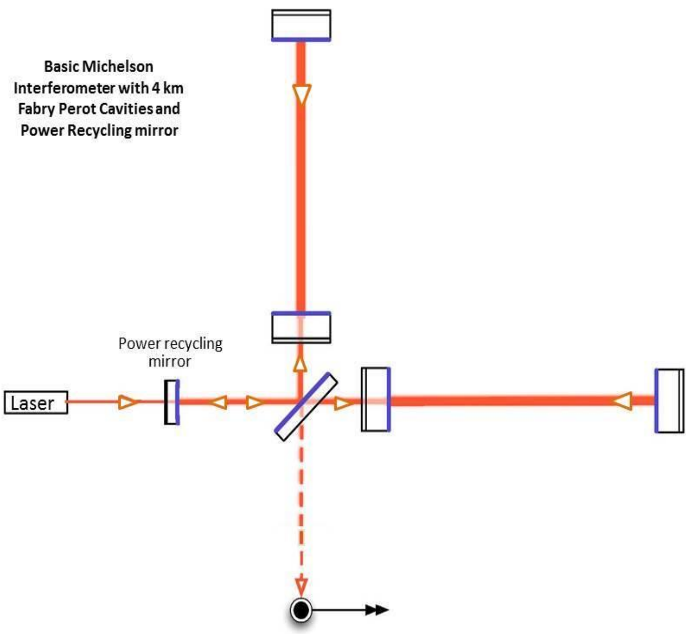
Michelson and LIGO's Interferometers

Although they are much more sophisticated, at their cores, LIGO's interferometers are fundamentally Michelson Interferometers, a device invented in the 1880's. We can say this because both Michelson and LIGO interferometers share these characteristics:

- They are both are L-shaped (not all interferometers are this shape)
- They both have mirrors at the ends of the arms to reflect light in order to combine light beams and create an interference pattern
- They both measure patterns and intensity of a resulting light beam after two beams have been superimposed or forced to 'interfere'

But this is where the similarities end. The size and added complexity of LIGO's interferometers are far beyond anything Michelson could have envisioned or that his original interferometer could have achieved.





Basic Michelson Interferometer with Fabry Perot cavities and Power Recycling mirror. Mirrors placed near thebeam splitter cause multiple reflections and keep the laser contained within the arms. This increases the distance traveled by the beams, greatly improving LIGO's sensitivity to changes in arm length like those caused by gravitational waves. LIGO's interferometers use multiple power recycling mirrors, but for simplicity only one is shown in the diagram.