Waves and Oscillations

Lecture No. 6

Topic: Wave Motion

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Wave Motion and Its Classification

- Wave motion is defined to be the disturbance created by the repeated periodic motion of the particles about their equilibrium position.
- Usually it is referred to as the transfer of energy from one point to another in a medium.
- Requirement of Medium divides it to 2 types-

1. Mechanical Wave (e.g. Sound)
2. Electromagnetic wave (e.g. Light)

Characteristics of mechanical wave

- It is produced by the periodic motion of the particles in the medium.
- Only the wave travels whereas the particles in the medium vibrates about their equilibrium positions.
- There is a regular change of phase between the various particles in the medium.
- The velocity of wave is different from the velocity of particle in the medium.

- Classification of waves in terms of propagation-

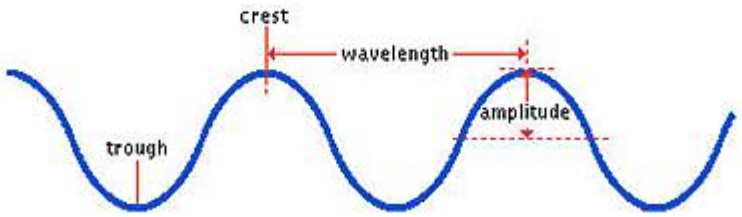
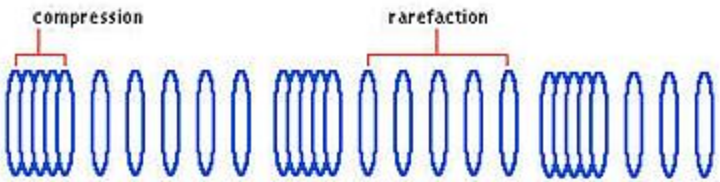
1. Travelling or progressive wave
2. Stationary or standing wave

- Classification of waves in terms of direction of vibration of particles-

1. Transverse wave
2. Longitudinal wave

Characteristics of travelling wave

- The particles in a medium vibrate periodically along the direction or perpendicular to the direction of propagation of wave.
- There is a gradual phase difference between the successive particles.
- Propagates with the formation of crest and trough or compression and rarefaction.
- Transfers energy while propagation.

Transverse waves	Longitudinal waves
1. Particles vibrate in a direction perpendicular to the direction of propagation of the wave.	1. Particles vibrate in a direction parallel to the direction of propagation of the wave.
2. Crests and troughs are formed.	2. Compressions and rarefactions are formed.
3. May be elastic or non-elastic wave. E.g. Ripples on the surface of water, wave on a string, light wave, radio wave, etc.	3. Only elastic wave (mechanical). E.g. Sound wave, seismic wave, etc.
4. Do not create pressure difference in the medium.	4. Create pressure difference in the medium.
5. They can be propagated through solids and surfaces of liquids but not in gases.	5. They can be propagated through solids, liquids and gases.
6. There is no change in the density of medium.	6. There is a change in the density throughout the medium.
	

Equation of a plane progressive or simple harmonic travelling wave

Let us consider a wave is originated at the point O and it travels towards right along X-axis. The equation of motion of the particle at O is,

$y = a\,\sin\omega t$ (7.1)

Here, y = displacement of particle at the time t

a = amplitude

ω = angular frequency

There must be a phase difference between any two points containing particles resides successive position. Let for a particle at point P which is at a distance x away from O the phase difference is ϕ. So the equation of motion of the particle at P is,

$y = a\,\sin(\omega t - \phi)$ (7.2)

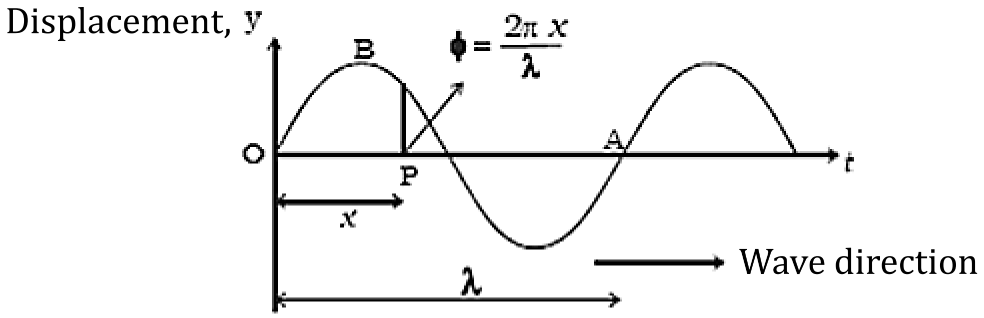


Figure: Travelling wave

For a path difference of one wavelength (λ) the corresponding phase difference is 2π. Hence, for a path difference x the corresponding phase difference will be,

$\phi = \frac{2\pi}{\lambda}x$ (7.3)

Substituting equation (7.3) in equation (7.2) we get

$y = a\,\sin(\omega t - \frac{2\pi}{\lambda}x)$ (7.4)

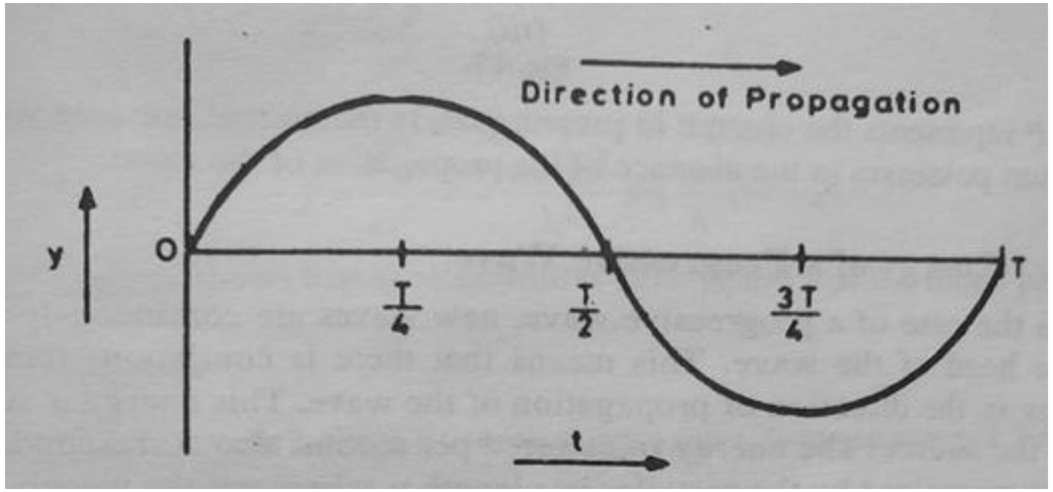
Or, $y = a\,\sin(\omega t - kx)$ (7.5)

The term, $k = \frac{2\pi}{\lambda}$ is the propagation constant or the angular wave number.

Now, angular frequency $\omega = \frac{2\pi}{T} = 2\pi n = \frac{2\pi v}{\lambda}$; [since, $n = \frac{v}{\lambda}$]

Equation (7.5) can be written as,

$y = a \sin(\frac{2\pi v}{\lambda} t - \frac{2\pi x}{\lambda})$



Or, $y = a \sin \frac{2\pi}{\lambda} (vt - x)$ (7.6)

Or, $y = a \sin k(vt - x)$ (7.7)

Or, $y = a \sin \frac{2\pi v}{\lambda} (t - \frac{x}{v})$ (7.8)

Or, $y = a \sin 2\pi n \, (t - \frac{x}{v})$ (7.9)

Or, $y = a \sin 2\pi \, (\frac{t}{T} - \frac{x}{\lambda})$ (7.10)

Similarly, if the wave travels towards left, x becomes negative and we have

$y = a \sin \frac{2\pi}{\lambda} (vt + x)$ (7.12)

Or, $y = a \sin(\omega t + kx)$ (7.13)

Differential equation of wave motion

The most general form of plane progressive wave or simple harmonic travelling wave equation is,

$$y = a \sin \frac{2\pi}{\lambda} (vt - x) \qquad (7.6)$$

Differentiating equation (7.6) with respect to time,

$$\frac{dy}{dt} = \frac{2\pi v a}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \qquad (7.14)$$

Now, differentiating equation (7.14) with respect to time,

$$\frac{d^2y}{dt^2} = -\frac{4\pi^2 v^2 a}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt - x)$$

$$\Rightarrow \frac{d^2y}{dt^2} = -\frac{4\pi^2 v^2}{\lambda^2} y \qquad (7.15)$$

Differentiating equation (7.6) with respect to *x*,

$$\frac{dy}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \qquad (7.16)$$

Now, differentiating equation (7.16) with respect to *x*,

$$\frac{d^2y}{dx^2} = -\frac{4\pi^2 a}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt - x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{4\pi^2}{\lambda^2} y \qquad (7.17)$$

Relating equations (7.15) and (7.17) we get,

$$\frac{d^2y}{dt^2} = v^2 \left(-\frac{4\pi^2}{\lambda^2} y\right)$$

$$\boxed{\Rightarrow \frac{d^2y}{dt^2} = v^2 \frac{d^2y}{dx^2}} \qquad (7.18)$$

Equation (7.18) is the differential equation of a plane progressive or travelling wave in one dimension.

The general differential equation of wave motion can also be written as,

$$\frac{d^2y}{dt^2} = K \frac{d^2y}{dx^2} \qquad (7.19)$$

Here, $K=v^2$ is a constant related to the compressibility of a medium.

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ rate of change of strain with respect to the distance travelled by the wave. Here, $\frac{dy}{dx}$ is known as the slope of the displacement curve.

So, **Particle acceleration at a point = (wave velocity)^2× slope of the strain curve.**

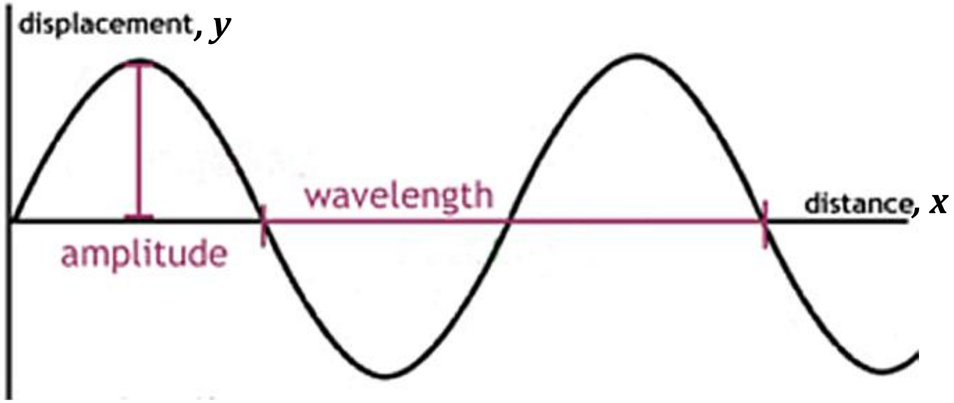


Figure: The displacement curve

- If we plot a strain ($\frac{dy}{dx}$) versus x curve the slope of that curve at any instant will give $\frac{d^2y}{dx^2}$

Some mathematical problems

- Analyze the following equations and find out which are the solutions of the one dimensional wave equation?

i. $y = x^2+v^2t^2$

ii. $y = x^2-v^2t^2$

iii. $y = 2x^2+v^2t^2$

iv. $y = (x-vt)^2$

v. $y = 2 \sin x \cos vt$

vi. $y = \sin 2x \cos vt$

vii. $y = \sin 3x \cos vt$

viii. $y = 2 \sin x \cos 4vt$

ix. $y = 15x - 8t$

x. $y = 5 \sin (12t - 7)$

Solutions:

i. $y = x^2+v^2t^2$

$\frac{dy}{dt} = 2v^2t$ and $\frac{d^2y}{dt^2} = 2v^2$

$\frac{dy}{dx} = 2x$ and $\frac{d^2y}{dx^2} = 2$

$Since, \frac{d^2y}{dt^2} = v^2 \times 2 = v^2 \frac{d^2y}{dx^2}$

$y = x^2+v^2t^2$ is a solution of one dimensional wave equation.

ii. $y = x^2-v^2t^2$

$\frac{dy}{dt} = -2v^2t$ and $\frac{d^2y}{dt^2} = -2v^2$

$\frac{dy}{dx} = 2x$ and $\frac{d^2y}{dx^2} = 2$

$Since, \frac{d^2y}{dt^2} = -v^2 \times 2 \neq v^2 \frac{d^2y}{dx^2}$

$y = x^2-v^2t^2$ is not a solution of one dimensional wave equation.

Some mathematical problems

1. The equation of displacement of a particle involved in a plane progressive wave motion at any instant of time is given by, $y = 0.25 \sin 2\pi(500t - 0.125x)$. Calculate the amplitude of the vibrating particle, wave velocity, wavelength, frequency, time period. Find the equations of particle velocity, particle acceleration and strain.

2. A plane progressive wave is travelling in a liquid medium. The wave travelling along positive X-direction with an amplitude = 3 cm, velocity = 180 m/s and frequency=300. Find out the displacement, velocity and acceleration of particle when the wave travels a distance 5 cm from the source after the time 6 s.

***Hints: identify *a, v, n, x* and *t* from the problem. Construct the equation of *y.* Differentiate *y* with respect to *t* once and twice to get particle velocity and acceleration, respectively.

3. A plane progressive wave of amplitude 8 cm is travelling along positive X-direction. At an instant of time, the displacement of a particle at a distance of 10 cm from the origin is +6 cm, meanwhile the displacement of another particle at a distance 25 cm from the origin is +4 cm. Evaluate the wavelength.

4. A sound wave with the frequency 512 Hz and amplitude 0.25 cm is generated from a source. The wave is propagating with the velocity 340 m/s in a medium with density 1.29×10^{-3} g/cm^3. Calculate the total energy per unit per unit volume and energy current per unit area of cross section during the wave motion?