

Waves and Oscillations

Lecture No. 1-3 (review)

Topics: Simple Harmonic Motion and Related derivations, examples

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Review of some earlier topics of the Course

Lectures: 1-5

- Differential equation of a simple harmonic oscillator
- Total energy and average energy
- Spring- mass system
- Calculation of time period of torsional pendulum
- Two-body oscillations, Reduced mass
- Combination of simple harmonic oscillations
- Lissajous figures.

Differential equation of a simple harmonic oscillator

Hook's law, $F=-ky$

F=restoring force, k=force constant

Newton's 2nd law of motion,

$$F = \text{mass} \times \text{acceleration} = m \frac{d^2y}{dt^2}$$

Combination of Hook's law and Newton's 2nd law of motion:

$$F = -ky = m \frac{d^2y}{dt^2}$$

$$\Rightarrow m \frac{d^2y}{dt^2} + ky = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \quad (1.1)$$

$$\omega = \sqrt{\frac{k}{m}} = \text{angular frequency}$$

$$\therefore \frac{d^2y}{dt^2} + \omega^2y = 0 \quad (1.2)$$

Equation (1.2) is the differential equation of a simple harmonic oscillator.

Solution:

$$\frac{d^2y}{dt^2} = -\omega^2y$$

$$\Rightarrow \int \left(2 \frac{dy}{dt} \right) \frac{d^2y}{dt^2} dt = -\omega^2 \int \left(2y \frac{dy}{dt} \right) dt$$

$$\Rightarrow \left(\frac{dy}{dt} \right)^2 = -\omega^2 y^2 + C \quad (1.3)$$

Applying boundary conditions, at $y_{\max} = a$, $\frac{dy}{dt} = 0$ [K.E. = 0]. So, from equation (1.3) we get, $C = \omega^2 a^2$

$$\text{Now, } \left(\frac{dy}{dt} \right)^2 = \omega^2 (a^2 - y^2)$$

$$\Rightarrow \frac{dy}{dt} = \pm \omega \sqrt{(a^2 - y^2)}$$

$$\Rightarrow \int \frac{dy}{\sqrt{(a^2 - y^2)}} = \omega \int dt$$

$$\Rightarrow \sin^{-1} \frac{y}{a} = \omega t + \varphi$$

$$\therefore y = a \sin(\omega t + \varphi) \quad (1.4)$$

Various Equations Related to SHM

Equations of Displacement

- $y = a \sin(\omega t + \varphi)$ (1.4)

- $y = a \sin \omega t \cos \varphi + a \cos \omega t \sin \varphi$ (1.5)

- $y = A \sin \omega t + B \cos \omega t$ (1.6)

- Velocity, $\frac{dy}{dt} = \omega \sqrt{(a^2 - y^2)}$ (1.7)

- Maximum velocity, $\left(\frac{dy}{dt}\right)_{\max} = \omega a$ (1.8)

- Acceleration, $\frac{d^2y}{dt^2} = -\omega^2 y$ (1.9)

- Maximum Acceleration, $\left(\frac{d^2y}{dt^2}\right)_{\max} = -\omega^2 a$ (1.10)

Assuming,
 $a \cos \varphi = A$
 $a \sin \varphi = B$

- Angular frequency, $\omega = \sqrt{\frac{k}{m}}$ (1.12)

- Time period, $T = 2\pi \sqrt{\frac{m}{k}}$ (1.13)

- Initial phase, φ

- If at $t=0, y=0$, then $\varphi=0$; (counting is started at the equilibrium) $y = a \sin \omega t$

- If $t=0, y=a$, then $\varphi=\pi/2$; (counting is started at the amplitude) $y = a \sin(\omega t + \pi/2) = a \cos \omega t$

- If $t=t', y=0$, (counting is started before the equilibrium position) then, $y = a \sin(\omega t - e)$

- If $t=t', y=0$, (counting is started after the equilibrium position) then, $y = a \sin(\omega t + e)$

Total energy of a particle executing SHM

- If no non-conservative forces (like friction) act on the oscillator the total energy of the particle will be,

$$E = K + U = \text{constant}$$

- Kinetic energy, $K = \frac{1}{2} m \left(\frac{dy}{dt} \right)^2 = \frac{1}{2} m [\omega a \cos(\omega t + \varphi)]^2$

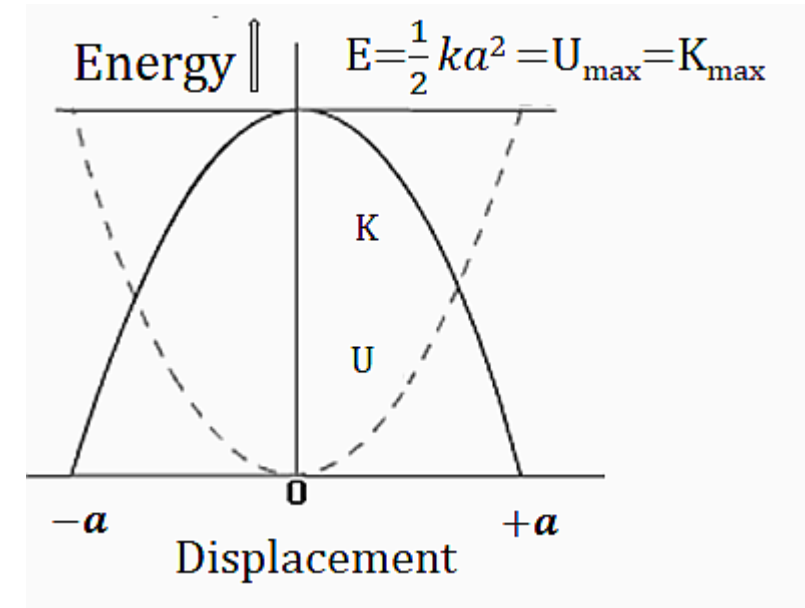
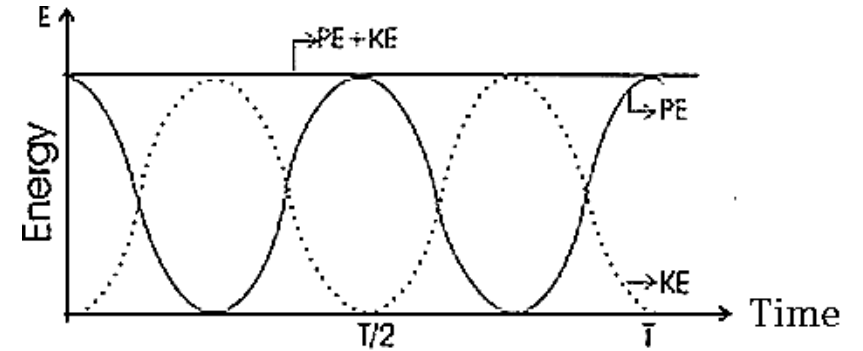
$$= \frac{1}{2} m \omega^2 a^2 \cos^2(\omega t + \varphi)$$

$$= \frac{1}{2} k a^2 \cos^2(\omega t + \varphi) \quad [\because m\omega^2 = k]$$
- Potential energy, $U = \int_0^y m \left(\frac{d^2y}{dt^2} \right) dy$

$$= \int_0^y m(\omega^2 y) dy = \frac{1}{2} m \omega^2 y^2$$

$$= \frac{1}{2} m \omega^2 a^2 \sin^2(\omega t + \varphi)$$

$$= \frac{1}{2} k a^2 \sin^2(\omega t + \varphi)$$
- $E = \frac{1}{2} k a^2 [\cos^2(\omega t + \varphi) + \sin^2(\omega t + \varphi)] = \frac{1}{2} k a^2 = \text{constant}$



Average energy of a particle executing SHM

- Average kinetic energy, $K_{\text{avg}} = \frac{1}{T} \int_0^T \frac{1}{2} ka^2 \cos^2(\omega t + \varphi) dt = \frac{ka^2}{4T} \int_0^T 2\cos^2(\omega t + \varphi) dt$
$$= \frac{ka^2}{4T} \int_0^T [1 + \cos 2(\omega t + \varphi)] dt$$
$$= \frac{ka^2}{4T} \left[\int_0^T dt + \int_0^T \cos 2(\omega t + \varphi) dt \right]$$
$$= \frac{ka^2}{4T} [t]_0^T + \frac{ka^2}{4T} \times 0$$
$$= \frac{1}{4} ka^2$$
- Average potential energy, $U_{\text{avg}} = \frac{1}{T} \int_0^T \frac{1}{2} ka^2 \sin^2(\omega t + \varphi) dt = \frac{ka^2}{4T} \int_0^T 2\sin^2(\omega t + \varphi) dt$
$$= \frac{ka^2}{4T} \int_0^T [1 - \cos 2(\omega t + \varphi)] dt$$
$$= \frac{ka^2}{4T} \left[\int_0^T dt - \int_0^T \cos 2(\omega t + \varphi) dt \right]$$
$$= \frac{ka^2}{4T} [t]_0^T - \frac{ka^2}{4T} \times 0$$
$$= \frac{1}{4} ka^2$$

Spring-mass System

Hook's law for extended spring, $F = -k\Delta l$

K =spring constant, Δl =extension, l =length of the spring [Fig (a)]

From Fig (b) Weight, $mg = k\Delta l$

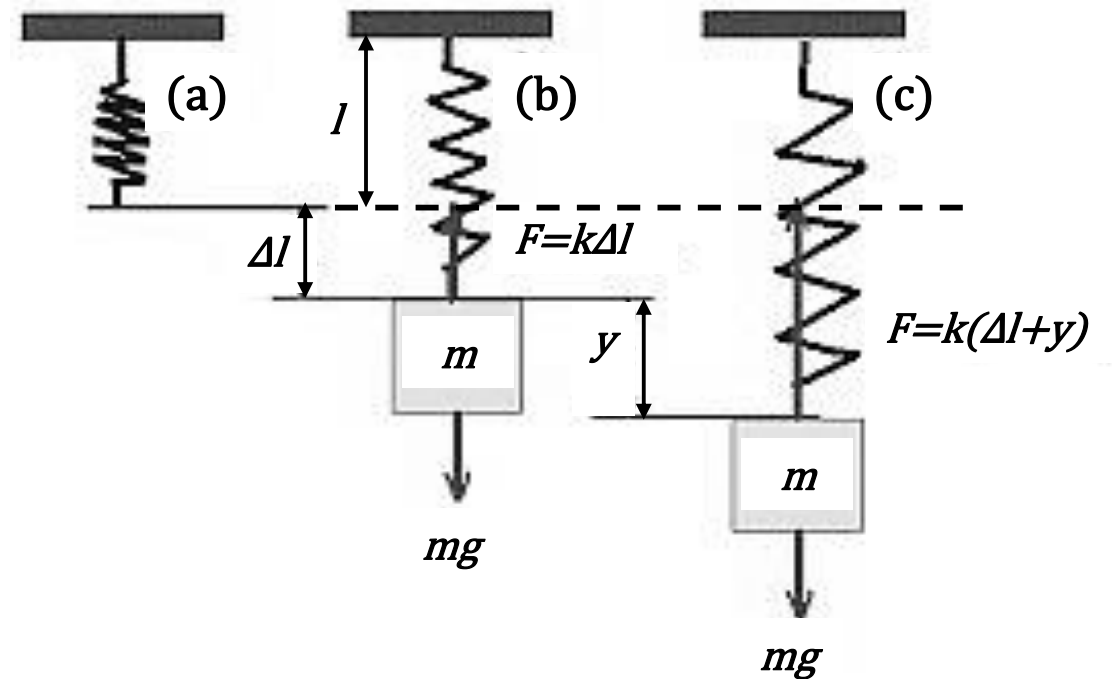
The Fig (c), The upward force the spring exerts on the body is $k(\Delta l + y)$

The downward force acting on the body is mg .

So, The resultant force on the body,

$$F = mg - k(\Delta l + y) = -ky$$

Newton's 2nd law of motion gives, $F = m \frac{d^2 y}{dt^2}$



Finally, $m \frac{d^2 y}{dt^2} = -ky$

$\frac{d^2 y}{dt^2} + \frac{k}{m} y = 0$; Same as the Diff. equation of SHM.

Time period, $T = 2\pi \sqrt{\frac{m}{k}}$

Torsional pendulum

Differential equation:

Hook's law for angular motion,

$$\tau = -\kappa\theta$$

κ =torsional spring constant

Newton's 2nd law for angular motion,

$$\tau = I\alpha = I \frac{d^2\theta}{dt^2}$$

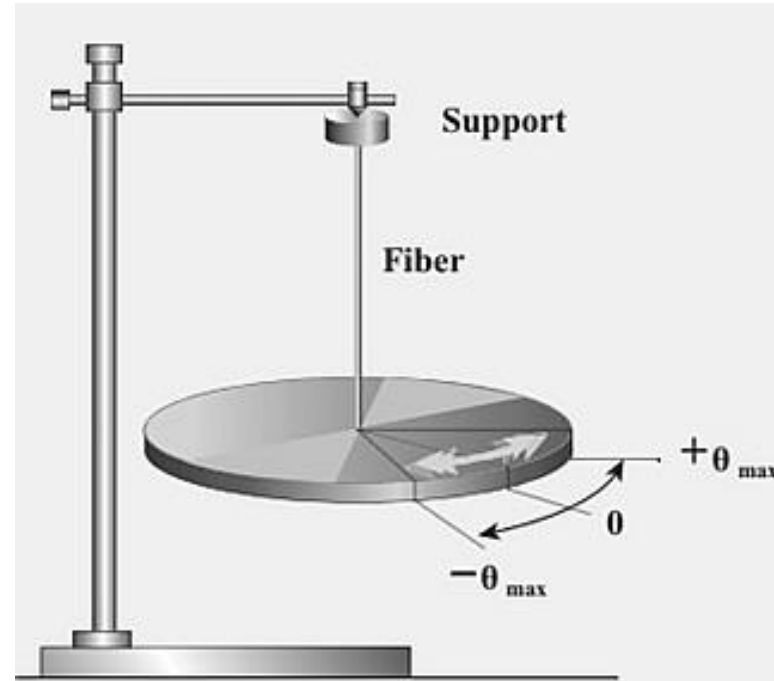
Equating expressions,

$$-\kappa\theta = I \frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{\kappa}{I}\theta = 0$$

Solution of the diff. equation

$$\theta = \theta_m \sin(\omega t + \varphi)$$



θ = angular displacement

θ_m = angular amplitude

ω = angular frequency = $\sqrt{\frac{\kappa}{I}}$

Time period, $T = 2\pi\sqrt{\frac{I}{\kappa}}$