Sound

Lecture No. 6

Topic: Wave Motion

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Particle velocity, wave velocity and particle acceleration

The most general form of plane progressive wave or simple harmonic travelling wave equation is,

$$y = a \sin \frac{2\pi}{\lambda}(vt - x) \qquad (7.6)$$

Here,

v = the velocity of the wave

y= displacement of the particle

Velocity of the particle, $U = \frac{dy}{dt}$

$$\frac{dy}{dt} = \frac{2\pi va}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \qquad (7.14)$$

The maximum value of the particle velocity is,

$$U_{max} = \left(\frac{dy}{dt}\right)_{\text{max}} = \frac{2\pi va}{\lambda} \qquad (7.20)$$

$\therefore$ **Maximum Particle Velocity =** $\boldsymbol{\frac{2\pi a}{\lambda}}$ **× Wave velocity**

Now, differentiating equation (7.6) with respect to x,

$$\frac{dy}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x) \qquad (7.16)$$

$$\therefore U = \frac{dy}{dt} = -v\frac{dy}{dx} \qquad (7.20)$$

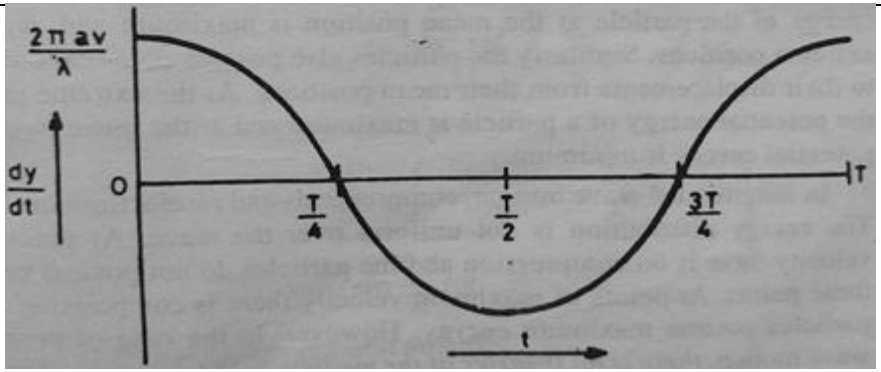
Particle velocity at any instant= Wave velocity × Slope of the displacement curve at that instant

Particle acceleration, f = $\frac{d^2y}{dt^2} = -\frac{4\pi^2v^2a}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x)$

$$\frac{d^2y}{dt^2} = -\frac{4\pi^2v^2}{\lambda^2}y \qquad (7.15)$$

The maximum value of the particle acceleration is,

$$\therefore f_{max} = \left(\frac{d^2y}{dt^2}\right)_{\text{max}} = -\left(\frac{4\pi^2v^2}{\lambda^2}\right)a \qquad (7.21)$$



Distribution of velocity and pressure in a travelling wave

The strain in the medium is,

$$\frac{dy}{dx} = -\frac{2\pi a}{\lambda}\cos\frac{2\pi}{\lambda}(vt - x) \qquad (7.16)$$

If $\frac{dy}{dx}$ is positive, it represents a region of rarefaction.

If $\frac{dy}{dx}$ is negative, it represents a region of compression.

The modulus of elasticity of the medium,

$$K = \frac{Change\ in\ pressure}{strain} = \frac{-dP}{\frac{dy}{dx}}$$

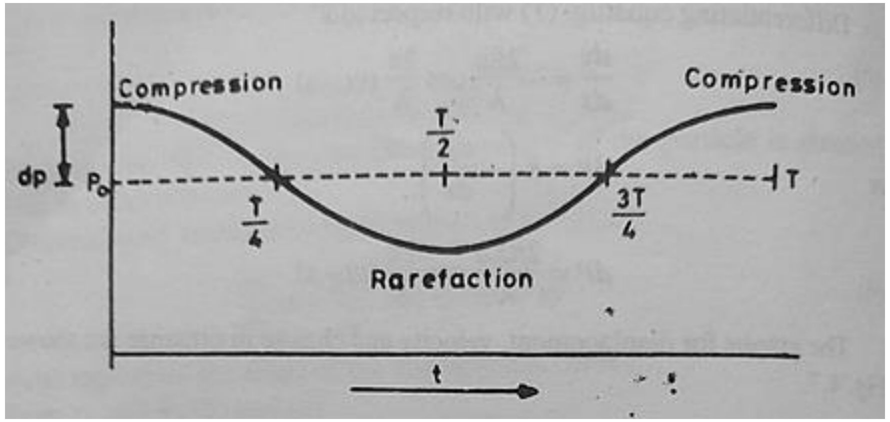
$$\therefore dP = -K\frac{dy}{dx}$$

$$\Rightarrow dP = K(-\frac{dy}{dx}) \Rightarrow dP = K(-\frac{dy}{dx})$$

$$\Rightarrow dP = \frac{2\pi Ka}{\lambda}\cos\frac{2\pi}{\lambda}(vt - x) \qquad (7.22)$$

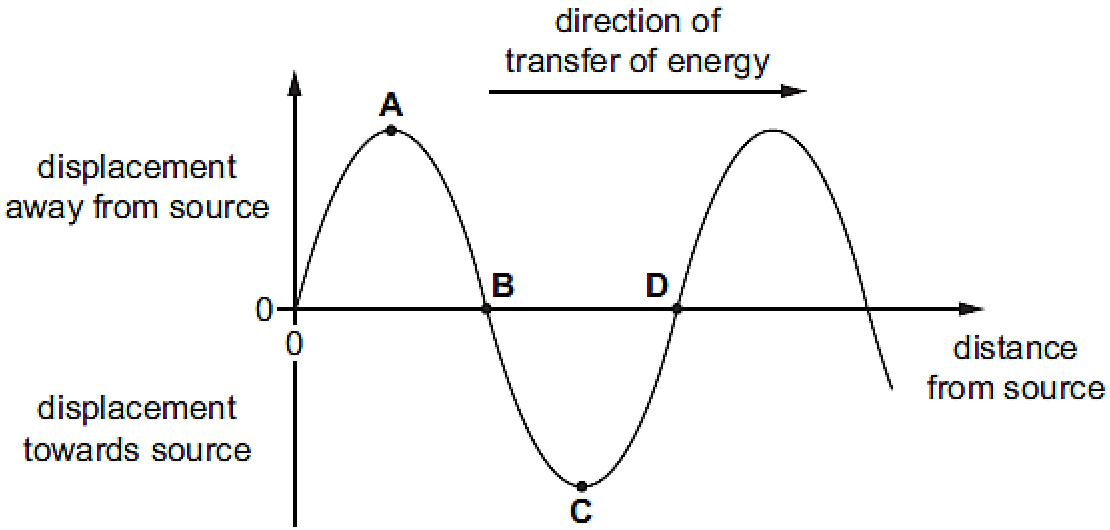
Equation (7.22) means that if $\frac{dy}{dx}$ is negative, dP is positive; it represents the a region of compression.

If $\frac{dy}{dx}$ is positive, dP is negative; it represents the a region of rarefaction.



This is a graph representing change in pressure with respect to time. Here, P_o is the normal pressure which the medium possesses in the absence of the propagation of wave.

Relation between particle displacement and wave propagation in a longitudinal travelling wave



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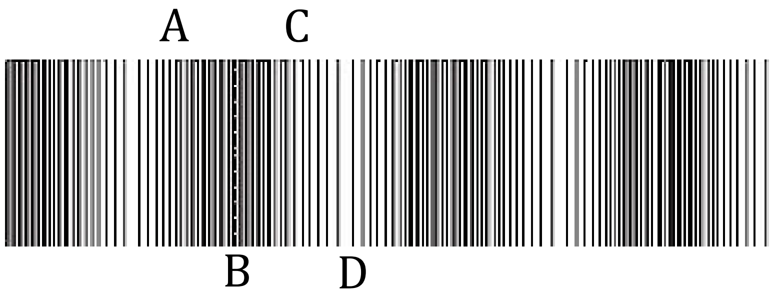
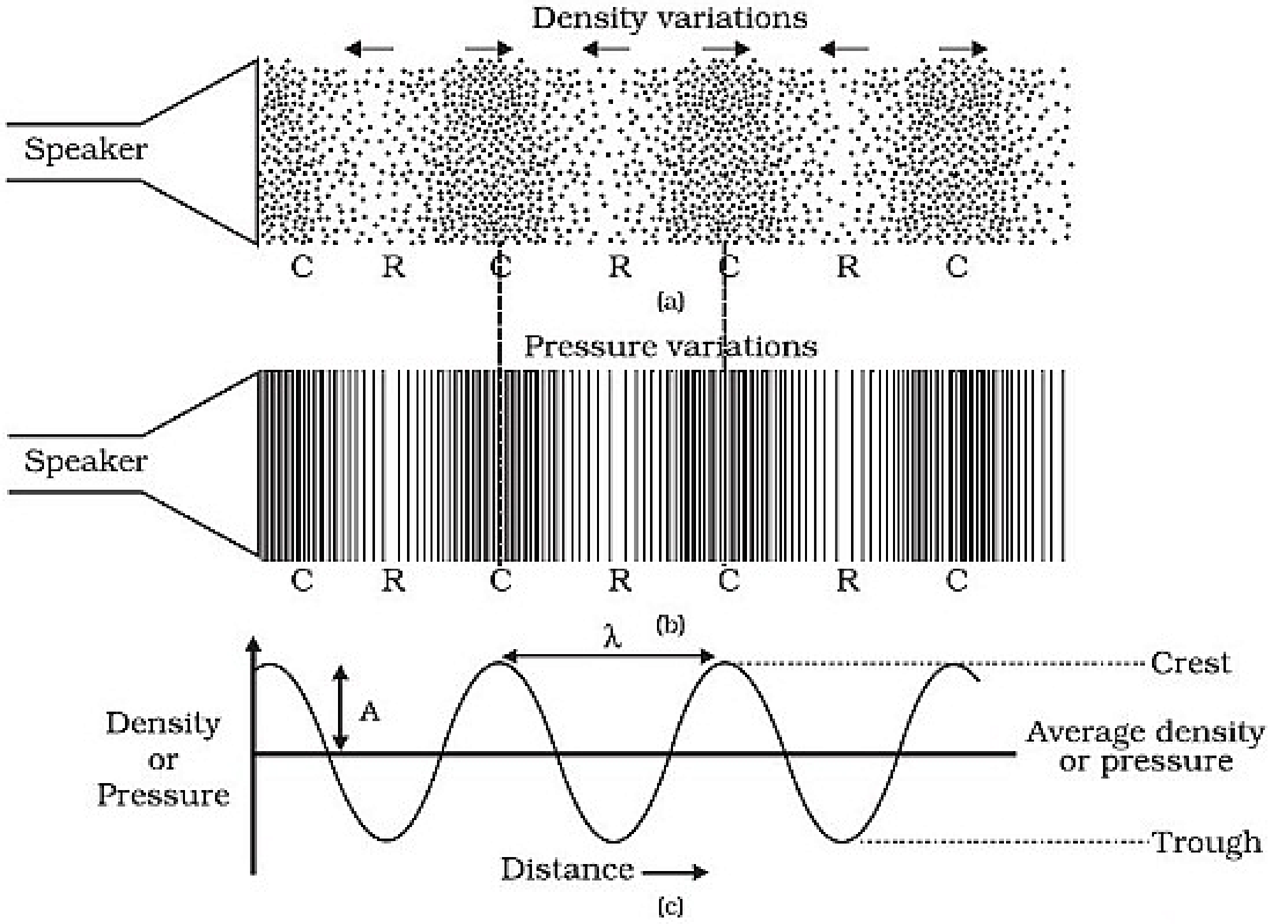


- From point O⟶A, particle moves away from the source and displacement is +ve.
- From point A⟶B, displacement is still +ve and particle is moving away from the source, but particle displacement is less.
- Point B is where displacement changes the sign from +ve to -ve, this is the center of compression.
- From point B⟶C, particle moves towards the source and displacement is -ve.
- From point C⟶D, displacement is still -ve and particle is moving towards the source, but particle displacement is less.
- Point D is where displacement changes the sign from -ve to +ve, this is the center of rarefaction.
- Points A and C are the midway between center of compression and rarefaction.

Distribution of density and pressure in a travelling longitudinal wave



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Energy of a travelling wave

- In a travelling wave, new waves are continuously generated at the head of the wave, that means there is a continuous transfer of energy in the direction of wave propagation.

- The amount of energy transferred per second is the energy possessed by the particle in a length v, where v is the wave velocity.

- The energy of a particle is partly kinetic (due to the velocity of the vibrating particle) and partly potential (due to the displacement of the particle from its equilibrium positions).

- Kinetic energy of a particle is maximum at the equilibrium position (velocity is maximum) and minimum at the amplitude.

- Potential energy of a particle is maximum at the amplitude and minimum at the equilibrium position.

- Since a longitudinal wave motion travels through creation of compression and rarefaction the energy distribution is not uniform over the wave.

- At the points of no velocity there is no compression and particles do not possess energy at these points. At the point of maximum velocity there is compression and particles possess maximum energy.

- Overall it can be said that in the case of travelling wave motion, <u>there is no transfer of the medium in the direction of propagation of the wave, but there is always transfer of energy in the direction of propagation of the wave.</u>

Energy of a travelling wave

Analytical treatment:

The equation simple harmonic travelling wave is,

$y = a \sin \frac{2\pi}{\lambda}(vt - x)$ (7.6)

Particle velocity at any instant,

$U = \frac{dy}{dt} = \frac{2\pi va}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x)$ (7.14)

Acceleration of the particle at any instant,

$f = \frac{d^2y}{dt^2} = \frac{dU}{dt} = -\frac{4\pi^2 v^2 a}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x) = -\frac{4\pi^2 v^2}{\lambda^2} y$ (7.15)

Potential energy

In order to move a particle from its equilibrium position to a distance, y an amount of work has to be done against acceleration. Let ρ is the density of the medium. Work done for a displacement dy is $Fdy = -m \frac{d^2y}{dt^2} dy$ [since work is done against f]

So, the work done per unit volume for a displacement dy is

$-\frac{m}{V} \frac{d^2y}{dt^2} dy = \rho \left[\frac{4\pi^2 v^2 a}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x)\right] dy$

Total work done for a displacement y is $= \rho \int_o^y \left[\frac{4\pi^2 v^2 a}{\lambda^2} \sin \frac{2\pi}{\lambda}(vt - x)\right] dy$

$= \frac{4\pi^2 v^2 \rho}{\lambda^2} \int_o^y y\,dy$

$= \frac{4\pi^2 \rho v^2 y^2}{2\lambda^2}$

P.E./vol $= \frac{2\pi^2 \rho v^2 y^2}{\lambda^2} = \frac{2\pi^2 \rho v^2 a^2}{\lambda^2} \sin^2 \frac{2\pi}{\lambda}(vt - x)$ (7.23)

Kinetic energy per unit volume, K.E./vol $= \frac{1}{2}\rho U^2$

$= \frac{1}{2}\rho \left[\frac{2\pi va}{\lambda} \cos \frac{2\pi}{\lambda}(vt - x)\right]^2$

$= \frac{1}{2}\rho \frac{4\pi^2 v^2 a^2}{\lambda^2} \cos^2 \frac{2\pi}{\lambda}(vt - x)$

K.E./vol $= \frac{2\pi^2 v^2 a^2 \rho}{\lambda^2} \cos^2 \frac{2\pi}{\lambda}(vt - x)$ (7.24)

Total energy per unit volume, E/vol = P.E./vol + K.E./vol

$= \frac{2\pi^2 v^2 a^2 \rho}{\lambda^2}\left[\sin^2 \frac{2\pi}{\lambda}(vt - x) + \cos^2 \frac{2\pi}{\lambda}(vt - x)\right]$

Since, $v = n\lambda$

E/vol $= \frac{2\pi^2 v^2 a^2 \rho}{\lambda^2} = 2\pi^2 n^2 a^2 \rho =$ constant (7.25)

Average P.E. and K.E. per unit volume

Here, Average P.E./vol = $\frac{1}{T}\int_o^T \frac{2\pi^2\rho\, v^2 a^2}{\lambda^2} sin^2\, \frac{2\pi}{\lambda}\ (vt - x)dt$

$$= \frac{2\pi^2\rho\, v^2 a^2}{2T\lambda^2}\int_0^T \left[1 - cos\,\frac{4\pi}{\lambda}\ (vt - x)\right]\ dt$$

$$= \frac{\pi^2\rho\, v^2 a^2}{T\lambda^2}[t]_0^T - \frac{\pi^2\rho\, v^2 a^2}{T\lambda^2} \times 0$$

$$= \frac{\pi^2\rho\, v^2 a^2 T}{T\lambda^2} = \pi^2 n^2 a^2 \rho \qquad (7.26)$$

Average K.E./vol = $\frac{1}{T}\int_o^T \frac{2\pi^2\rho\, v^2 a^2}{\lambda^2} cos^2\, \frac{2\pi}{\lambda}\ (vt - x)dt$

$$= \frac{2\pi^2\rho\, v^2 a^2}{2T\lambda^2}\int_0^T \left[1 + cos\,\frac{4\pi}{\lambda}\ (vt - x)\right]\ dt$$

$$= \frac{\pi^2\rho\, v^2 a^2}{T\lambda^2}[t]_0^T + \frac{\pi^2\rho\, v^2 a^2}{T\lambda^2} \times 0 = \frac{\pi^2\rho\, v^2 a^2 T}{T\lambda^2} = \pi^2 n^2 a^2 \rho \qquad (7.27)$$

So, E/vol = Average P.E./vol + Average K.E./vol = $2\pi^2 n^2 a^2 \rho$ The potential energy and kinetic energies of every particle change with time, but Average P.E./vol and Average K.E./vol remain constant. The total energy per unit volume remains constant.

Intensity of wave

Let the area of cross section of a parallel beam of radiation is 1 unit and the velocity of the wave is v.

Volume = $1 \times v = v$

Energy transfer per unit area per second,

I = (E/vol)$\times v$

I = $2\pi^2 n^2 a^2 \rho\, v$ (7.28)

Energy transfer per second is also called energy current per unit area of cross section or current density per unit area or **intensity of wave**. The current density per unit volume remains constant.

Intensity of wave is defined as the rate of transfer of energy per unit area and the area being perpendicular to the direction of propagation of wave.

Power of wave motion

The time-averaged power (P_{ave}) of a sinusoidal mechanical wave is the average rate of energy transfer associated with a wave as it passes a point. It can be found by taking the total energy associated with the wave divided by the time it takes to transfer the energy.

If the velocity of the sinusoidal wave (v) is constant, the time for one wavelength to pass by a point is equal to the period of the wave, which is also constant. For a sinusoidal mechanical wave, the time-averaged power is therefore the energy associated with a wavelength divided by the period of the wave (T).

The amount of energy associated with a wavelength is = (E/vol)×λ $= 2\pi^2 n^2 a^2 \rho\, \lambda$

$$P_{ave} = \frac{2\pi^2 n^2 a^2 \rho\, \lambda}{T} = 2\pi^2 n^2 a^2 \rho\, n\lambda$$

$\Rightarrow P_{ave} = 2\pi^2 n^2 a^2 \rho\, v$ (7.29) (dimensionally same as the intensity)

$P_{ave} = ½\, (2\pi n)^2 a^2 \rho\, v$

Since, $\omega = 2n\pi$,

$P_{ave} = ½\, \omega^2 a^2 \rho\, v$ (7.30)