Waves and Oscillations

Lecture No. 2

Topics:

Energy of a Simple Harmonic Oscillator, Examples of SHM, Two-body oscillation, Reduced Mass

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Total energy of a particle executing SHM

- If no non-conservative forces (like friction) act on the oscillator the total energy of the particle will be,

  E= K+U=constant

- Kinetic energy, $K=\frac{1}{2}m\left(\frac{dy}{dt}\right)^2=\frac{1}{2}m[\omega a cos(\omega t+\varphi)]^2$

  $=\frac{1}{2}m\,\omega^2 a^2 cos^2(\omega t+\varphi)$

  $=\frac{1}{2}ka^2 cos^2(\omega t+\varphi)$ $[\because m\omega^2=k]$

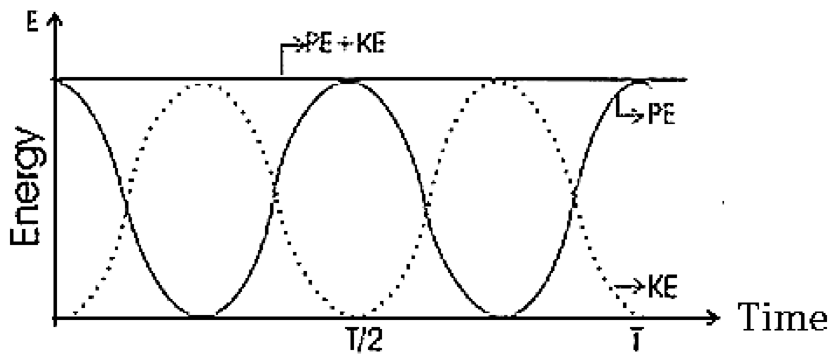
- Potential energy, $U=-\int_0^y m\left(\frac{d^2y}{dt^2}\right)dy$

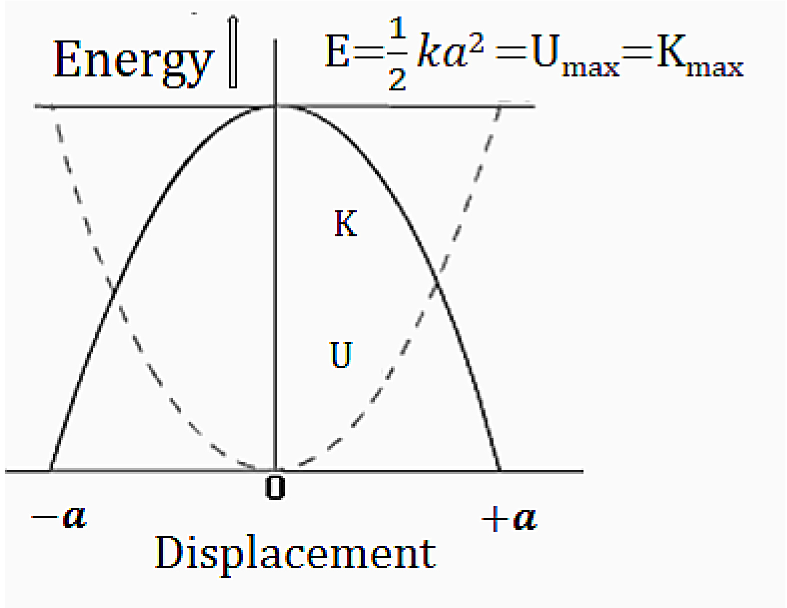
  $=-\int_0^y m(-\omega^2 y)\,dy=\frac{1}{2}m\omega^2 y^2$

  $=\frac{1}{2}m\,\omega^2 a^2 sin^2(\omega t+\varphi)$

  $=\frac{1}{2}ka^2 sin^2(\omega t+\varphi)$

- $E=\frac{1}{2}ka^2[cos^2(\omega t+\varphi)+sin^2(\omega t+\varphi)]=\frac{1}{2}ka^2=$ constant





Average energy of a particle executing SHM

- Average kinetic energy, K_{avg} $=\frac{1}{T}\int_0^T \frac{1}{2} ka^2cos^2(\omega t+\varphi)\ dt = \frac{ka^2}{4T}\int_0^T 2cos^2(\omega t+\varphi)\ dt$

$=\frac{ka^2}{4T}\int_0^T [1+cos2(\omega t+\varphi)]\ dt$

$=\frac{ka^2}{4T}\left[\int_0^T dt + \int_0^T cos2(\omega t+\varphi) dt\right]$

$=\frac{ka^2}{4T}[t]_0^T + \frac{ka^2}{4T}\times 0$

$=\frac{1}{4}ka^2$

- Average potential energy, U_{avg} $=\frac{1}{T}\int_0^T \frac{1}{2} ka^2sin^2(\omega t+\varphi)\ dt = \frac{ka^2}{4T}\int_0^T 2sin^2(\omega t+\varphi)\ dt$

$=\frac{ka^2}{4T}\int_0^T [1-cos2(\omega t+\varphi)]\ dt$

$=\frac{ka^2}{4T}\left[\int_0^T dt - \int_0^T cos2(\omega t+\varphi) dt\right]$

$=\frac{ka^2}{4T}[t]_0^T - \frac{ka^2}{4T}\times 0$

$=\frac{1}{4}ka^2$

$\int_0^T cos2(\omega t+\varphi) dt$

$= \frac{1}{2\omega}[sin\ 2(\omega t+\varphi)]_0^T$

$= \frac{1}{2\omega}[sin\ 2(\omega T+\varphi) - sin2\varphi]$

$= \frac{1}{2\omega}[sin\ 2\left(\omega \frac{2\pi}{\omega}+\varphi\right) - sin2\varphi]$

$= \frac{1}{2\omega}[sin\ 2(2\pi+\varphi) - sin2\varphi]$

$= \frac{1}{2\omega}[sin2\varphi - sin2\varphi]$

$= 0$

Spring-mass System

Hooke's law for extended spring, $F=-k\Delta l$

K=spring constant, Δl=extension, l=length of the spring [Fig (a)]

From Fig (b)Weight, $mg=k\Delta l$

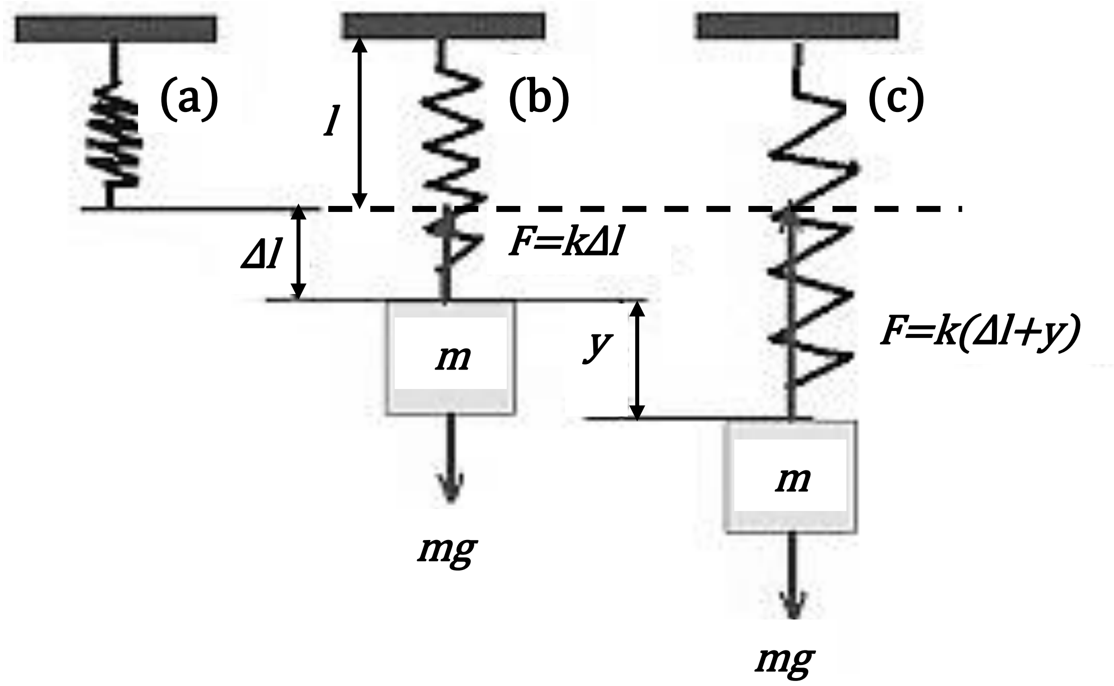
The Fig (c), The upward force the spring exerts on the body is $k(\Delta l+y)$

The downward force acting on the body is mg.

So, The resultant force on the body,

$F=mg-k(\Delta l+y)=-ky$

Newton's 2nd law of motion gives, $F=m\frac{d^2y}{dt^2}$



Finally, $m\frac{d^2y}{dt^2}=-ky$

$\frac{d^2y}{dt^2}+\frac{k}{m}y=0$; Same as the Diff. equation of SHM.

Time period, $T=2\pi\sqrt{\frac{m}{k}}$

Torsional pendulum

Differential equation:

Hooke's law for angular motion,

$$\tau = -\kappa\theta$$

κ=torsional spring constant

Newton's 2nd law for angular motion,

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2}$$

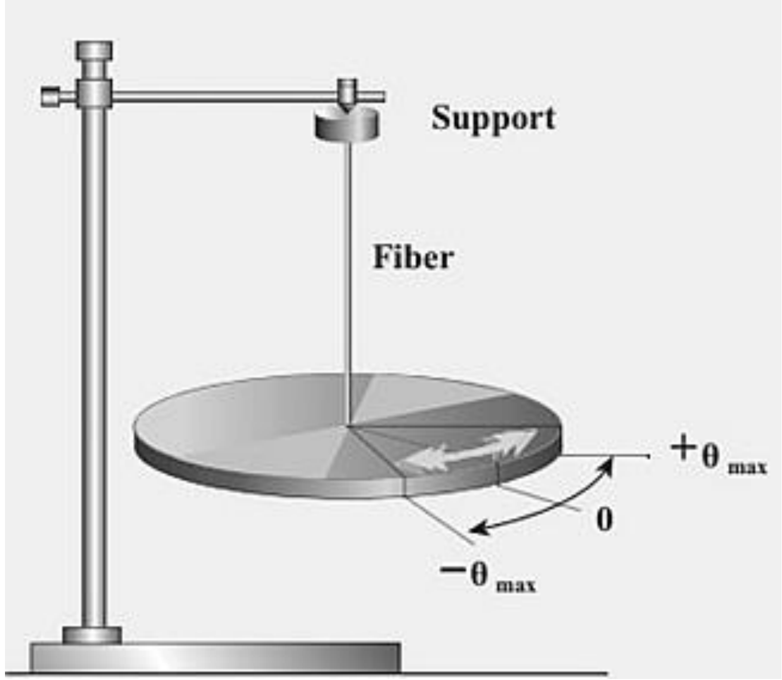
Equating expressions,

$$-\kappa\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{\kappa}{I}\theta = 0$$

Solution of the diff. equation

$$\theta = \theta_m sin(\omega t + \varphi)$$



θ = angular displacement

θ_m = angular amplitude

ω = angular frequency = $\sqrt{\frac{\kappa}{I}}$

Time period, T = $2\pi\sqrt{\frac{I}{\kappa}}$

Simple Harmonic oscillation in LC circuit

Simple harmonic oscillation in an electrical system

- The capacitor (C) gets charged upon pressing key S.
- C discharges through the inductance coil on releasing S.
- Magnetic flux (ϕ) increases due to the current, $I=\frac{dQ}{dt}$
- ϕ induces emf ($-L\frac{dI}{dt}$), which opposes the growth of current.
- L opposes both growth and decay of current in the circuit.
- Voltage drop across the capacitor = $\frac{Q}{C}$

Since there is no external emf in the circuit (the battery being cut-off), the net emf in the circuit is,

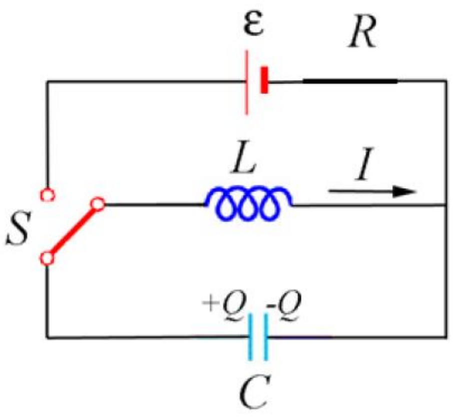
$\frac{Q}{C} + L\frac{dI}{dt}=0$ (From Kirchhoff's law)

$\Rightarrow \frac{Q}{LC} + \frac{dI}{dt}=0$

$\Rightarrow \frac{Q}{LC} + \frac{d^2Q}{dt^2}=0$

$\Rightarrow \frac{d^2Q}{dt^2}+ \omega^2 Q = 0$ (Where, $\omega^2 = \frac{1}{LC}$ or, $\omega = \frac{1}{\sqrt{LC}}$)

This equation is similar to the differential equation of SHM with y replaced by Q, m replaced by L and k replaced by $\frac{1}{C}$.



ε = emf of the source
R =resistance
S= switch
L = inductance
C =capacitance
Q= charge on each plate of C

Hence, the solution can be written as,

$Q= Q_o \sin(\omega t+\varphi)$

Q_o = amplitude of charge

Frequency of variation of charge between $+Q_o$ to $-Q_o$ is,

$n= \frac{1}{2\pi\sqrt{LC}}$

Time period, $T= 2\pi\sqrt{LC}$

$I= \frac{dQ}{dt}= Q_o \omega \cos(\omega t+\varphi)$;

Since maximum value of $\cos(\omega t+\varphi)=1$

Maximum current, $I_o=Q_o\omega$

$I=I_o \cos(\omega t+\varphi)$

Two-Body Oscillations

- In microscopic world, many objects such as nuclei, atoms, molecules, etc. execute oscillations that are approximately SHM.

- Example: Diatomic molecule in which 2 atoms are bonded together with a force. Above absolute zero temperature, the atoms vibrate continuously about their equilibrium positions.

- We can compare such a molecule with a system where the atoms can be considered as two particles with different masses connected by a spring.

Let the molecules can be represented by two masses m_1 and m_2 connected to each other by a spring of force constant k as shown in Fig 4(a).

The motion of the system can be described in terms of the separate motions of the two particles which are located relative to the origin O by the two coordinates x_1 and x_2 in Fig. 4(a).

The relative separation $(x_1 - x_2)$ gives the length of the spring at any time.

The un-stretched length of the spring is L.

The change in length of the spring is given by,

$$x = (x_1 - x_2) - L \qquad (4.1)$$

The magnitude of the force that the spring exerts on each particle is,

$$F = kx \qquad (4.2)$$

If the spring exerts a force $-\vec{F}$ on m_1, then it exerts a force $\vec{F}$ on m_2.

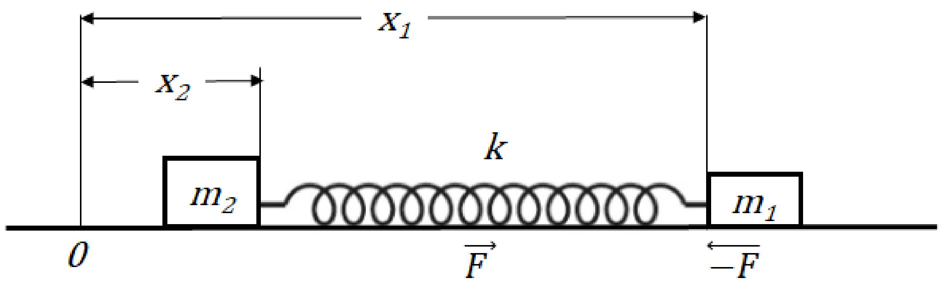


Fig. 4(a)

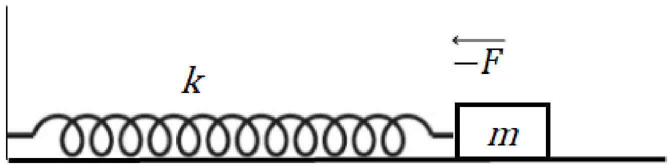


Fig. 4(b)

Taking the force component along the X-axis, let us apply Newton's 2nd law of motion separately to the two particles,

$$m_1 \frac{d^2 x_1}{dt^2} = -kx \qquad (4.3)$$

$$m_2 \frac{d^2 x_2}{dt^2} = kx \qquad (4.4)$$

Multiplying equation (4.3) by m_2 and equation (4.4) by m_1

$$m_1 m_2 \frac{d^2 x_1}{dt^2} = -m_2 kx \quad (4.5)$$

$$m_1 m_2 \frac{d^2 x_2}{dt^2} = m_1 kx \quad (4.6)$$

Subtracting, equation (4.6) from equation (4.5),

$$m_1 m_2 \frac{d^2 x_1}{dt^2} - m_1 m_2 \frac{d^2 x_2}{dt^2} = -m_2 kx - m_1 kx$$

$$\Rightarrow m_1 \, m_2 \frac{d^2}{dt^2}(x_1 - x_2) = -kx(m_1 + m_2)$$

$$\Rightarrow \frac{m_1 m_2}{(m_1 + m_2)} \frac{d^2}{dt^2}(x_1 - x_2) = -kx \quad (4.7)$$

The quantity $\frac{m_1 m_2}{(m_1 + m_2)}$ has the dimension of mass. This quantity is known as the reduced mass of the system and it is denoted by μ.

$$\mu = \frac{m_1 m_2}{(m_1 + m_2)} \quad (4.8)$$

Reduced mass of a system is always smaller than either of the masses of the system. ($\mu < m_1$ and $\mu < m_2$)

Since, the un-stretched length of the spring is constant the derivative of (x_1-x_2) are the same as the derivative of x.

$$\frac{d^2}{dt^2}(x_1 - x_2) = \frac{d^2}{dt^2}(x + L) = \frac{d^2 x}{dt^2} \text{ [from equation (4.1)]}$$

So, from equation (4.7) we get,

$$\mu \frac{d^2 x}{dt^2} = -kx$$

$$\Rightarrow \frac{d^2 x}{dt^2} + \frac{k}{\mu} x = 0 \quad (4.9)$$

Here, angular frequency is, $\omega = \sqrt{\frac{k}{\mu}}$; So, time period, $T = 2\pi\sqrt{\frac{\mu}{k}}$

Equation (4.9) is identical to the differential equation of SHM of a single body oscillator.