Waves and Oscillations

Lecture No. 5

Topics: Forced oscillations and Resonance

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Forced Oscillation

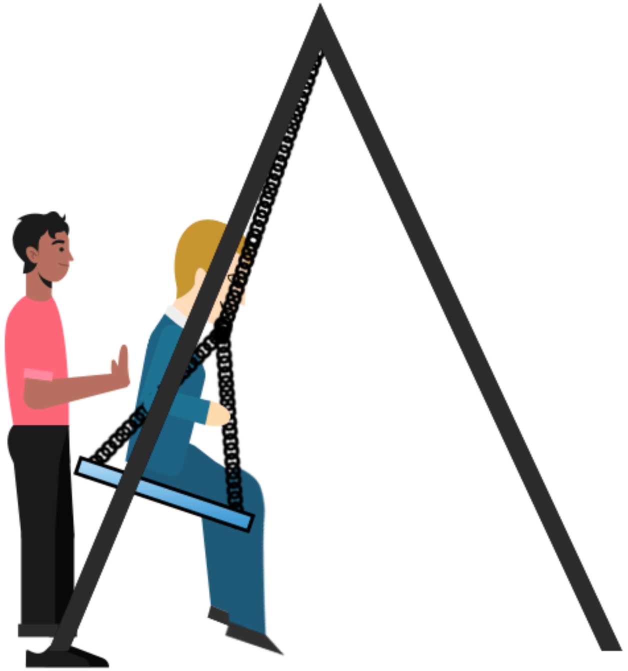
- The time period of a simple harmonic oscillator depends on the dimensions of the body and its elastic properties.

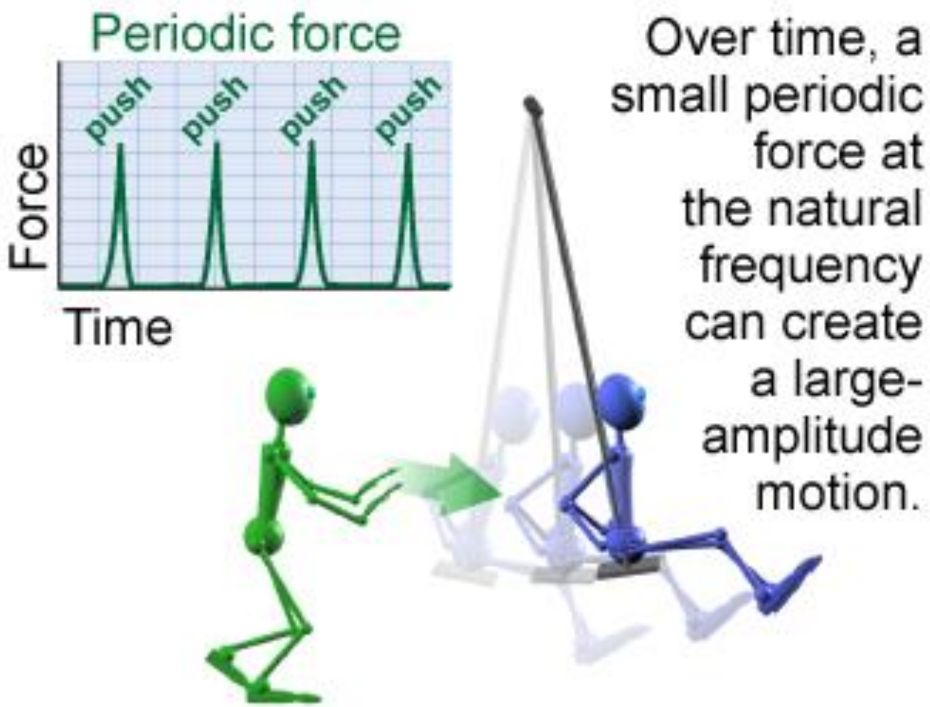
- The vibration of such body die out with time due to the dissipation of energy. (Damping)

- If some external periodic force is constantly applied on the body, it continues to oscillate under the influence of such external force. Such vibration of the body are called **FORCED VIBRATIONS**.

Examples of Forced Oscillation

- Motion of a swing, musical instruments, stringed instruments, etc.





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Differential equation of a forced oscillator

If damping is taken into consideration for an oscillator, then oscillator experiences

(i) Restoring Force : $F_r = -ky$;

(ii) Damping Force : $F_d = -b\frac{dy}{dt}$;

(iii) Let an external force is applied to the damped oscillator which given by, $F_e = F_o e^{iqt}$

(iv) We, therefore, can write the equation of the forced oscillation as, $F = F_d + F_r + F_e$

Combination of Hook's law and Newton's 2nd law of motion gives us, $m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} + F_o e^{iqt}$

$$m\frac{d^2y}{dt^2} + ky + b\frac{dy}{dt} = F_o e^{iqt}$$

$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{F_o}{m}e^{iqt}$$

$$\frac{d^2y}{dt^2} + 2p\frac{dy}{dt} + \omega^2 y = f\,e^{iqt} \qquad (6.1)$$

Equation (6.1) is a 2^{nd} order 1^{st} degree differential equation for forced vibration.

Where, $f = \frac{F_o}{m}$ is the amplitude of driving force per unit mass.

Solution:

Let us consider the trials solution of equation (6.1),

$y=Ae^{iqt}$ (6.2)

Or, $\frac{dy}{dt} = Aiqe^{iqt}$

Or, $\frac{d^2y}{dt^2} = -q^2Ae^{iqt}$

Using these values in equation (6.1),

$-q^2Ae^{iqt} + 2p\,Aiqe^{iqt} + \omega^2Ae^{iqt} = f\,e^{iqt}$

Or, $A[-q^2 + 2ipq + \omega^2] = f$

Or, $A = \frac{f}{(\omega^2 - q^2) + i2pq}$ (6.3)

Let, $(\omega^2 - q^2) = B\cos\varphi$ and $2pq = B\sin\varphi$

$B^2 = B^2\cos^2\varphi + B^2\sin^2\varphi$

$= 4p^2q^2 + (\omega^2 - q^2)^2$

So, $B = \sqrt{4p^2q^2 + (\omega^2 - q^2)^2}$

$\tan\varphi = \frac{B\sin\varphi}{B\cos\varphi} = \frac{2pq}{(\omega^2 - q^2)}$

Substituting these values in the equation (6.3),

$A = \frac{f}{B(\cos\varphi + i\sin\varphi)} = \frac{f}{Be^{i\varphi}}$

So, $A = \frac{fe^{-i\varphi}}{\sqrt{4p^2q^2 + (\omega^2 - q^2)^2}}$

Substituting A in equation (6.2),

$y = \frac{f}{\sqrt{4p^2q^2 + (\omega^2 - q^2)^2}}\,e^{i(qt-\varphi)}$ (6.4)

Equation(6.4) represents a SHM with the angular frequency q (same as the external or driving force). The forced SHM will be lagging behind the force by a phase φ.

Now, equation (6.1) is an inhomogeneous differential equation. Hence, $y=Ae^{iqt}$ is not really a complete solution.

The solution will be complete if a complementary function is added which is a solution of the related homogeneous equation,

$$\frac{d^2y}{dt^2}+2p\frac{dy}{dt}+\omega^2 y=0$$

Here, applying the boundary condition another solution can be obtained when $F_o=0$.

This corresponds to the oscillatory motion in presence of damping for which the solution is,

$$y=ae^{-pt}cos\left[\sqrt{(\omega^2-p^2)}t-\gamma\right]$$

Here, a and γ are the constants depending on the initial condition.

So, the general solution of the equation (6.1) is,

$$y=ae^{-pt}cos\left[\sqrt{(\omega^2-p^2)}t-\gamma\right]+\frac{f}{\sqrt{4p^2q^2+(\omega^2-q^2)^2}}e^{i(qt-\varphi)} \qquad (6.5)$$

The 1st part of the solution represent the initial damped oscillation with the angular frequency $\sqrt{(\omega^2-p^2)}$ and the amplitude decaying exponentially to zero. The 2nd part of the solution represents the forced vibration with the angular frequency q and the constant amplitude A.

Resonance

- When the forced frequency is equal to the natural frequency of oscillator the oscillation will have maximum amplitude and the state of oscillation of a system is called **RESONANCE.** The amplitude of a forced oscillator is $A = \frac{f}{\sqrt{4p^2q^2 + (\omega^2 - q^2)^2}}$
- *"A"* will be maximum when the denominator is the minimum. That means when,

$$\frac{d}{dq}[4p^2q^2 + (\omega^2 - q^2)^2] = 0$$

Or, $8p^2q + 2(\omega^2 - q^2)(-2q) = 0$

Or, $-4q[(\omega^2 - q^2) - 2p^2] = 0$

Or, $(\omega^2 - q^2) - 2p^2 = 0$ [since $q \neq 0$]

Or, $q^2 = \omega^2 - 2p^2$

$\therefore q = \sqrt{\omega^2 - 2p^2} \quad = \sqrt{\omega^2 - \frac{b^2}{2m^2}}$ [since, $\omega^2 > 2p^2$ and $2p = b/m$]

The resonance frequency, $f_R = \frac{\sqrt{\omega^2 - 2p^2}}{2\pi}$, for which A is the maximum.

Sharpness of resonance:

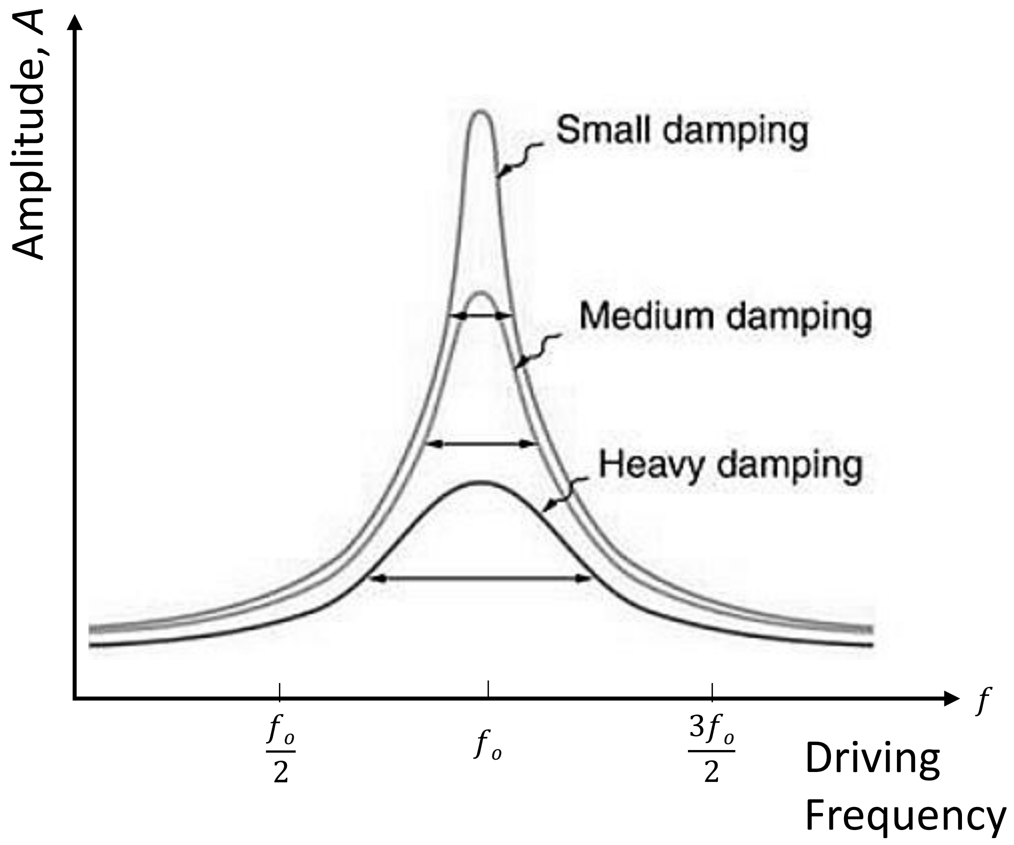
Sharpness of resonance is referred to the fall in amplitude with the change in frequency on each side of the maximum amplitude.

The amplitude of a forced oscillator is $A = \frac{f}{\sqrt{4p^2q^2 + (\omega^2 - q^2)^2}}$

(i) For small damping, damping constant p is low and $q = \omega$, $A_{max} = \frac{f}{2pq}$. The resonance curve is sharper.

(ii) For large damping p is high and resonance curve is flat.

(iii) In absence of any damping force $p=0$, amplitude of resonance is infinite.



Response:

The particular solution for displacement in the case of forced oscillation is,

$$y = \frac{f}{\sqrt{4p^2q^2 + (\omega^2 - q^2)^2}} e^{i(qt-\varphi)}$$

Differentiating equation of y with respect to time we get,

$$\frac{dy}{dt} = \frac{fq}{\sqrt{4p^2q^2 + (\omega^2 - q^2)^2}} e^{i(qt-\varphi)}$$

The velocity is maximum when $e^{i(qt-\varphi)}$ is the maximum, that is the oscillator crosses the equilibrium position. So,

$$\left(\frac{dy}{dt}\right)_{max} = \frac{fq}{\sqrt{4p^2q^2 + (\omega^2 - q^2)^2}}$$

Kinetic energy of the oscillator at the instant of crossing the equilibrium position is given by,

$$K = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2_{max} = \frac{\frac{1}{2}mf^2q^2}{4p^2q^2 + (\omega^2 - q^2)^2}$$

The mean square of the driving force per unit mass$= \frac{0+f^2}{2} = \frac{f^2}{2}$

Response, $R = \frac{K}{\frac{f^2}{2}} = \frac{\frac{1}{2}mf^2q^2}{\left[4p^2q^2 + (\omega^2 - q^2)^2\right]\left[\frac{f^2}{2m}\right]} = \frac{mq^2}{\left[4p^2q^2 + (\omega^2 - q^2)^2\right]}$

$$R = \frac{mq^2}{\left[4p^2q^2 + (\omega^2 - q^2)^2\right]}$$

R will be maximum when, $q = \omega$; $R = \frac{mq^2}{4p^2q^2} = \frac{m}{4p^2}$

Since, $2p = b/m$, $R = \frac{m^3}{b^2}$

$R \propto \frac{1}{b^2}$; Response is inversely proportional to square of damping co-efficient of the medium. In absence of damping R is maximum.

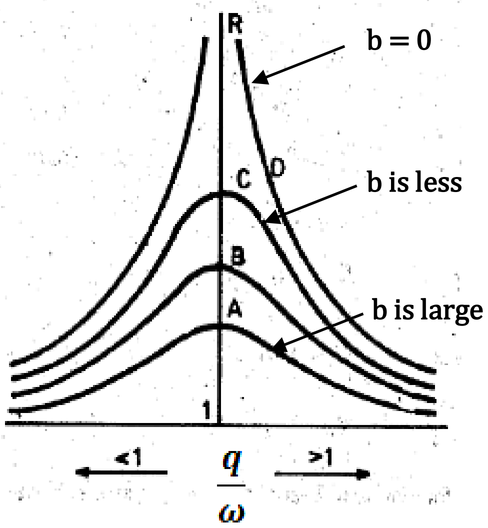
i. $\frac{q}{\omega} = 1$, R is maximum

ii. When b=0, R is infinite,

Sharpness of resonance is maximum.

iii. Sharpness of resonance decreases with increase of b.

iv. Sharpness of resonance dies very rapidly even for a very small change in the value of $\frac{q}{\omega}$ from 1, where b is minimum.



Phase of resonance:

Considering the phase of the forced SHM,

$$tan\varphi = \frac{2pq}{(\omega^2 - q^2)}$$

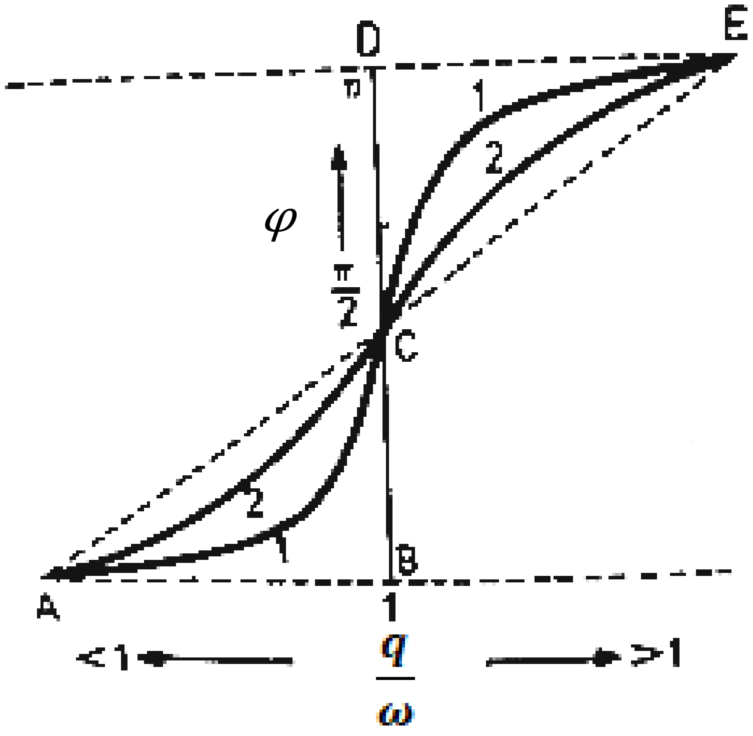
At resonance $\omega^2 = q^2$ and $tan\varphi = \infty$ That is, $\varphi = \pi/2$

Thus for $\frac{q}{\omega} = 1$, $\varphi = \frac{\pi}{2}$

For $\frac{q}{\omega} > 1$, $\varphi > \frac{\pi}{2}$

For $\frac{q}{\omega} < 1$, $\varphi < \frac{\pi}{2}$

The shape of the curve will also depend on the value of *b* that is the external frictional forces.



i. For *b=0*, Curve ABCDE

ii. For *b* very small , Curve 1

iii. For *b* very large , Curve 2