Sound

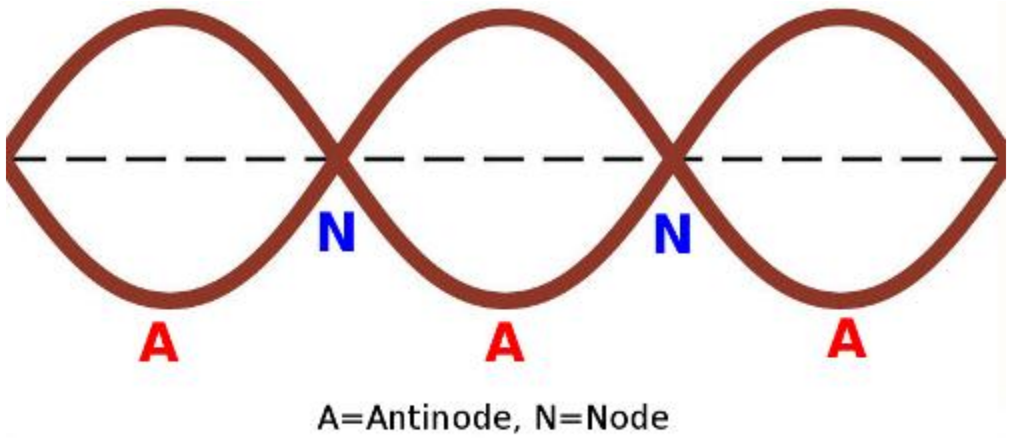
Lecture No. 7

Topic: Stationary Wave

Teacher’s name: Dr. Mehnaz Sharmin

Stationary waves

Definition: Stationary or standing wave is one in which crests and troughs (for transverse) or compressions and rarefactions (for longitudinal) do not change their location in space.



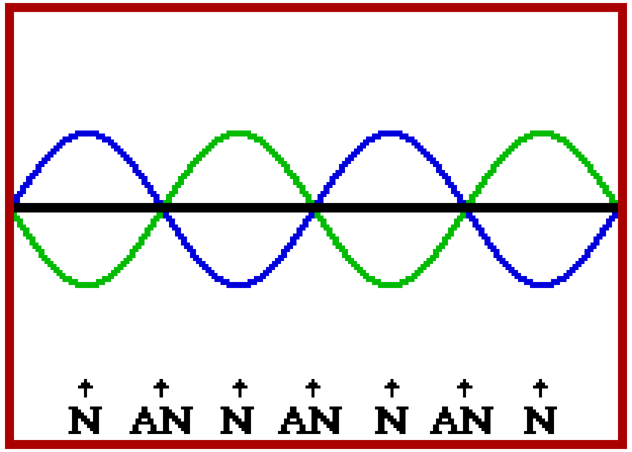


Figure: Stationary waves

- Stationary waves are produced when two progressive waves, having the same amplitudes and periods superimpose while travelling in opposite directions with the same velocity.
- In the figure, the points "N" are called nodes where the amplitude of the oscillation is zero. The points "A" are called antinodes where the amplitude of the particles is maximum.

Characteristics of stationary waves

- The stationary waves are formed because of the superposition of a wave and its reflected wave.
- This wave does not appear to travel in space, They do not propagate energy.
- All particles except at the nodes vibrate simple harmonically with time period equal to that of each component wave.
- Particles on the either side of the node vibrate in opposite phase and those Particles on the either side of the antinode vibrate in the same phase.
- The amplitude of vibration of the particles gradually increases between zero and maximum from node to antinode.
- The whole medium is split into segments and all the particles of the same segment vibrate in phase. The particles in one segment have a phase difference of π with those in the neighboring segment.
- For a longitudinal stationary wave the pressure (density) variations are maximum at the node and minimum at the antinode.
- Amplitude of vibration of the particles is a function of position and phase of vibration of particles is a function of time.
- Strain is maximum at the nodes and minimum at the antinode. Energy associated with the vibration is maximum at the antinode and minimum at the node.

Interference of sound wave

The equations of two waves are as follows-

$y_1 = a \sin\frac{2\pi}{\lambda}(vt-x)$ (9.1)

$y_2 = b \sin\frac{2\pi}{\lambda}(vt-x+\varphi)$ (9.2)

When they meet each other from the opposite directions, the resultant wave equation becomes

$y = y_1 + y_2 = a \sin\frac{2\pi}{\lambda}(vt-x) + b \sin\frac{2\pi}{\lambda}(vt-x+\varphi)$

$= a\sin\frac{2\pi}{\lambda}(vt-x) + b\sin\frac{2\pi}{\lambda}(vt-x)\cos\varphi + b\cos\frac{2\pi}{\lambda}(vt-x)\sin\varphi$

$= [\sin\frac{2\pi}{\lambda}(vt-x)]\,(a + b\cos\varphi) + [\cos\frac{2\pi}{\lambda}(vt-x)](b\sin\varphi)$

Let, $a + b\cos\varphi = A\cos\theta$

and $b\sin\varphi = A\sin\theta$

$\therefore A = \sqrt{A^2\cos^2\theta + A^2\sin^2\theta} = \sqrt{a^2 + b^2 + 2ab\cos\varphi}$

and $\theta = \tan^{-1}\frac{b\sin\varphi}{a + b\cos\varphi}$

$\therefore y = [\sin\frac{2\pi}{\lambda}(vt-x)]\,(A\cos\theta) + [\cos\frac{2\pi}{\lambda}(vt-x)](A\sin\theta)$

$= A \sin\frac{2\pi}{\lambda}[(vt-x) + \theta]$ (9.3)

<u>Special cases</u>

i. When phase difference $\varphi = 0, 2\pi, 4\pi, \ldots\ldots\ldots = 2n\pi$; *where,* $n = 0, 1, 2, \ldots\ldots$

$\cos\varphi = 1$

$A = \sqrt{a^2 + b^2 + 2ab} = a + b$ (Amplitude is maximum)

This is the case of constructive interference where two waves reinforce each other.

If $a = b$, $A = 2a$, intense sound is heard.

ii. When $\varphi = \pi, 3\pi, 5\pi \ldots\ldots = (2n+1)\pi$; *where,* $n = 0, 1, 2, \ldots\ldots$

$\cos\varphi = -1$

$A = \sqrt{a^2 + b^2 - 2ab} = a - b$ (Amplitude is minimum)

This is the case of destructive interference and feeble sound will be produced.

If $a = b$, $A = 0$, no sound is heard.

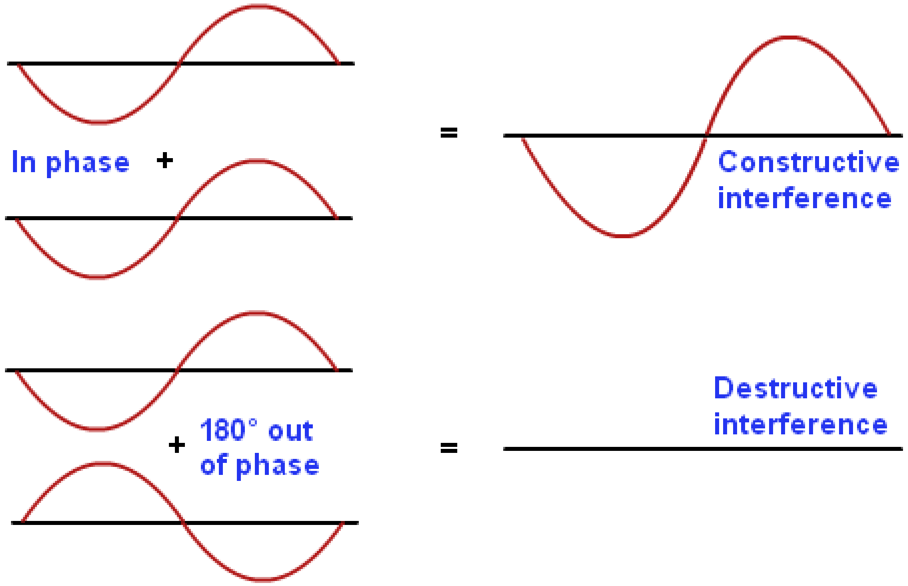


Figure: Interference of sound wave

Intensity of resultant wave is

$$I \propto \left(\sqrt{a^2+b^2+2abcos\varphi}\right)^2 = a^2+b^2+2abcos\varphi$$

If $a=b$,

$$I \propto a^2+a^2+2a^2cos\varphi = 2a^2(1+cos\varphi)$$

$$= 2a^2 \times 2cos^2\varphi/2$$

$$= 4a^2 cos^2\varphi/2$$

For constructive interference: $I_{max}=4a^2$

For destructive interference: $I_{min}=0$

Beats

The equations of two waves with slightly different frequencies, travelling along the same path in the same direction are as follows-

$y_1 = a \sin\omega_1 t$ (9.4) $[\omega_1 = 2\pi n_1]$

$y_2 = b \sin \omega_2 t$ (9.5) $[\omega_2 = 2\pi n_2]$

According to the principle of superposition,

$y = y_1 + y_2 = a \sin \omega_1 t + b \sin \omega_2 t$

$= a \sin \omega_1 t + b \sin [\omega_1 - (\omega_1 - \omega_2)]t$

$= a \sin \omega_1 t + b \sin\omega_1 t \cos(\omega_1 - \omega_2)t - b \cos\omega_1 t \sin(\omega_1 - \omega_2)t$

$= \sin \omega_1 t [a + b \cos(\omega_1 - \omega_2)t] - \cos\omega_1 t [b \sin(\omega_1 - \omega_2)t]$

Let, $a + b \cos(\omega_1 - \omega_2)t = A \cos\theta$

and $b \sin(\omega_1 - \omega_2)t = A \sin\theta$

$\therefore y = \sin \omega_1 t \, A \cos\theta - \cos \omega_1 t \, A \sin\theta$

$y = A \sin (\omega_1 t - \theta)$ (9.6)

Resultant amplitude:

$A^2 \cos^2\theta + A^2 \sin^2\theta = [a + b \cos(\omega_1 - \omega_2)t]^2$

$+ [b \sin(\omega_1 - \omega_2)t]^2$

Or, $A^2 = a^2 + b^2 \cos^2 (\omega_1 - \omega_2)t + 2ab\cos(\omega_1 - \omega_2)t + b^2 \sin^2 (\omega_1 - \omega_2)t$

Or, $A^2 = a^2 + b^2 + 2ab\cos(\omega_1 - \omega_2)t$

$\therefore A = \sqrt{a^2 + b^2 + 2ab\cos(\omega_1 - \omega_2)t}$ (9.7)

Phase angle of the resultant wave:

$\tan\theta = \frac{A\sin\theta}{A\cos\theta} = \frac{b \sin(\omega_1 - \omega_2)t}{a + b \cos(\omega_1 - \omega_2)t}$

$\therefore \theta = \tan^{-1} \frac{b \sin(\omega_1 - \omega_2)t}{a + b \cos(\omega_1 - \omega_2)t}$ (9.8)

Amplitude and phase angle of the resultant wave both changes with time.

Case-I

When $(\omega_1\text{-}\omega_2)t = 2\pi(n_1\text{-}n_2)\ t = 2k\pi$; where $k=0, 1, 2, \ldots\ldots$

The resultant amplitude is then

$$A = \sqrt{a^2 + b^2 + 2ab} = \sqrt{(a+b)2} = (a+b)$$

Thus the resultant amplitude is maximum. Since, intensity ∝ (amplitude)^2 the intensity of sound will be maximum.

When t = $\frac{2k\pi}{2\pi(n_1-n_2)} = \frac{k}{(n_1-n_2)}$

That is, at the time instant 0, $\frac{1}{(n_1-n_2)}, \frac{2}{(n_1-n_2)}, \ldots\ldots\ldots\ldots\ldots$ the maximum intensity sound will be heard.

Case-II

When $(\omega_1\text{-}\omega_2)t = 2\pi(n_1\text{-}n_2)\ t = (2k+1)\pi$; where $k=0, 1, 2, \ldots\ldots$

The resultant amplitude is then

$$A = \sqrt{a^2 + b^2 - 2ab} = \sqrt{(a-b)2} = (a\text{-}b)$$

Thus the resultant amplitude is minimum.

When t = $\frac{(2k+1)\pi}{2\pi(n_1-n_2)} = \frac{(2k+1)}{2(n_1-n_2)}$

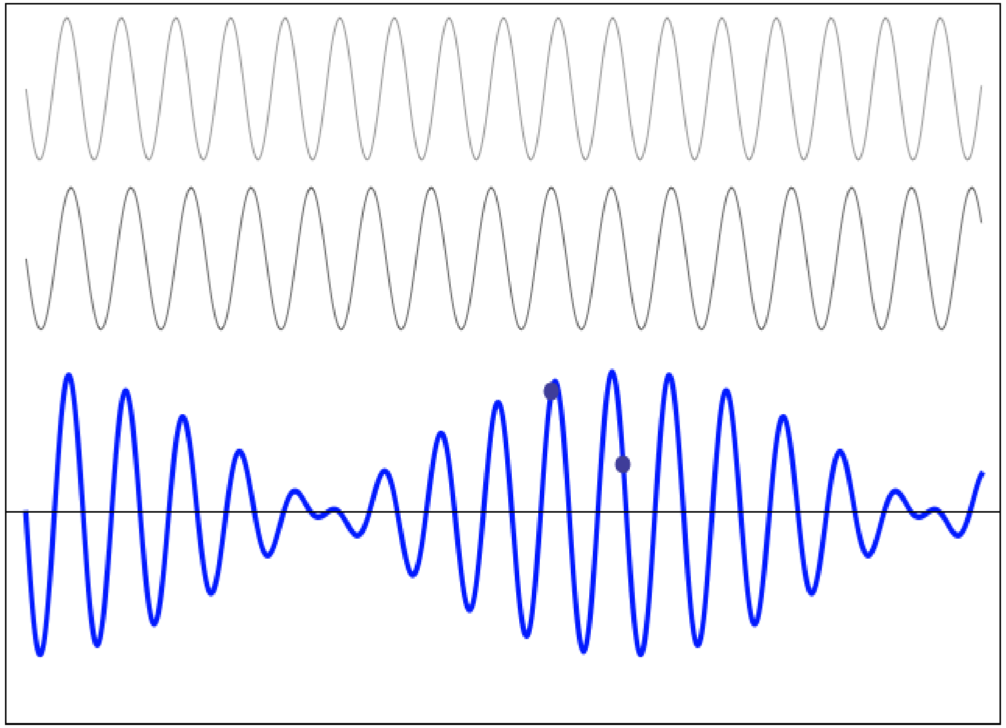
That is, at the time instant $\frac{1}{2(n_1-n_2)}, \frac{3}{2(n_1-n_2)}, \ldots\ldots\ldots\ldots\ldots$ the minimum intensity sound will be heard.

Thus the time interval between successive maxima and minima is $\frac{1}{(n_1-n_2)}$ sec. One minimum amplitude is produced between two successive maxima and vice versa.

Hence, the number of beats produced per second = $\frac{1}{1/(n_1-n_2)}$

$= (n_1\text{-}n_2)$

Thus the number of beats produced per second is equal to the difference in frequency of the two notes.



Phase velocity and group velocity

Let us consider the superposition occurs between two waves with nearly same frequency, follow the equations,

$y_1 = a\, sin(\omega_1 t - k_1 x)$ (9.9)

$y_2 = a\, sin(\omega_2 t - k_2 x)$ (9.10)

The equation of the resultant wave is,

$y = y_1 + y_2 = a\, sin(\omega_1 t - k_1 x) + a\, sin(\omega_2 t - k_2 x)$

Applying the trigonometric relation,

$sinA + sinB = 2\, sin\frac{1}{2}(A+B)\, cos\frac{1}{2}(A-B)$

So, $y = 2a\, sin\, [\frac{(\omega_1+\omega_2)}{2}\, t - \frac{(k_1+k_2)}{2}\, x]\, cos[\frac{(\omega_1-\omega_2)}{2}\, t - \frac{(k_1-k_2)}{2}\, x]$

Let us consider, $\omega_1 - \omega_2 = \Delta\omega$ and $k_1 - k_2 = \Delta k$; where $\Delta\omega$ and Δk are very small.

Also, $\omega = \frac{1}{2}(\omega_1+\omega_2)$ and $k = \frac{1}{2}(k_1 + k_2)$

$y = 2a\, cos\, \frac{1}{2}(\Delta\omega.t - \Delta k.x)\, sin\, (\omega t - k.x)$ (9.11)

Thus, the resultant wave has the same frequency and wavelength as the original with the amplitude modulated by a factor $cos\, \frac{1}{2}(\Delta\omega.t - \Delta k.x)$.

The velocity of the resultant wave, $v = \omega/k$ is nearly equal to that of the individual waves and that is called the phase velocity.

The envelope gives the cosine wave and is called the **Beat wave**. It consists of a group of waves, each group consists of a number of waves. **Group velocity** is nothing but the velocity at which the envelope travels.

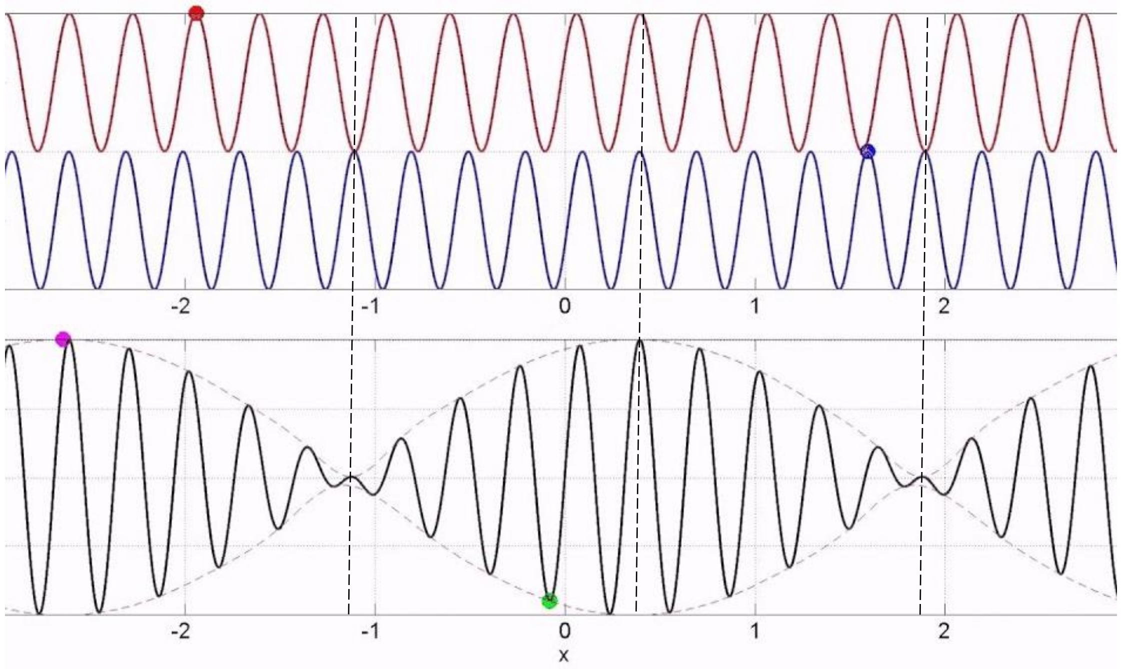


Figure: Phase velocity and group velocity

Relation between phase velocity and group velocity

The envelope travels as a wave with the wave number $\frac{1}{2}\Delta k$ and angular frequency $\frac{1}{2}\Delta\omega$.

The velocity at which the envelope travels can be determined by considering the modulating factor, $cos\,\frac{1}{2}(\Delta\omega.t - \Delta k.x)$

$=cos\,(\frac{1}{2}\Delta\omega.t - \frac{1}{2}\Delta k.x)$

$= cos\,\frac{1}{2}\Delta k\,(x - \frac{\Delta\omega}{\Delta k}\,.\,t)$

$= cos\,\frac{1}{2}\Delta k\,(x - ut)$

Where, $u=\frac{\Delta\omega}{\Delta k}$ is called the group velocity.

But, $\omega= vk$ and v is the phase velocity.

Hence the relation between group velocity (u) and phase velocity (v) is given by,

$u = \frac{\Delta\omega}{\Delta k} = \frac{\Delta(vk)}{\Delta k}$

$Or,\ u = v + k\frac{\Delta v}{\Delta k}$

$Or,\ u = v + k\frac{\Delta v}{\Delta\lambda}\frac{\Delta\lambda}{\Delta k}$

Since $k=\frac{2\pi}{\lambda}$ or, $\lambda=\frac{2\pi}{k}$

Hence, $\frac{\Delta\lambda}{\Delta k} = -\frac{2\pi}{k^2}$

Thus, $u = v - k\frac{\Delta v}{\Delta\lambda}\frac{2\pi}{k^2} = v - \frac{2\pi}{k}\frac{\Delta v}{\Delta\lambda}$

$\therefore u = v - \lambda\frac{\Delta v}{\Delta\lambda}$ (9.12)

- A medium in which the phase velocity does not depend upon the frequency of the wave, then u=v, such medium is called non-dispersive medium. Examples: Waves on a perfectly flexible string, sound wave in air, light waves in vacuum, etc.

- If the phase velocity depends on the frequency or wavelength, then $u \neq v$, the medium is dispersive in nature. Examples: Water waves, light waves in water, etc.