Waves and Oscillations

Lecture No. 4

Topic: Damped Oscillation

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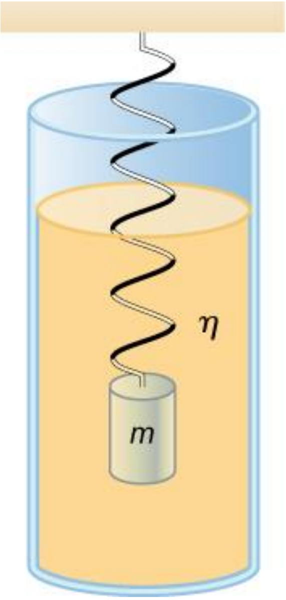
Free Oscillation and Damped Oscillation

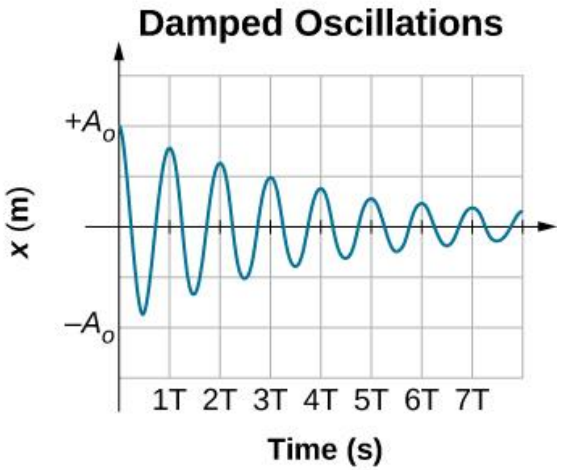
- If a oscillation occurs flawlessly without any resistive force acting on it is called free oscillation.
- Any oscillation occurring in an air medium, experiences frictional force and consequent energy dissipation occurs.
- The amplitude of oscillation decays continuously with time and finally diminishes. Such oscillation is called damped oscillation.
- The dissipated energy appears as heat either within the oscillating system itself or in the surrounding medium.

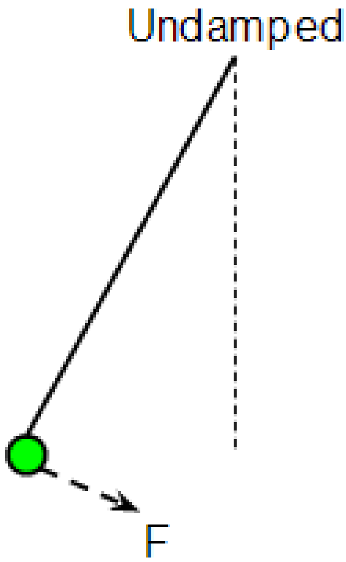
Characteristics of Damped Oscillation

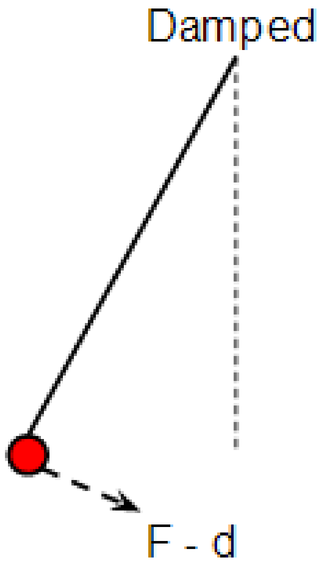
- Frictional force acting on a body opposite to the direction of its motion is called damping force.
- Damping force reduces the velocity and the kinetic energy of the moving body.
- Damping or dissipative forces generally arises due to the viscosity or friction in the medium and are non-conservative in nature.
- When velocities of body are not high, damping force is found to be proportional to velocity (v) of the particle
- The frequency of damped oscillator is always less than that of it's natural or undamped frequency.
- Amplitude of oscillation does not remain constant, rather it decays with time

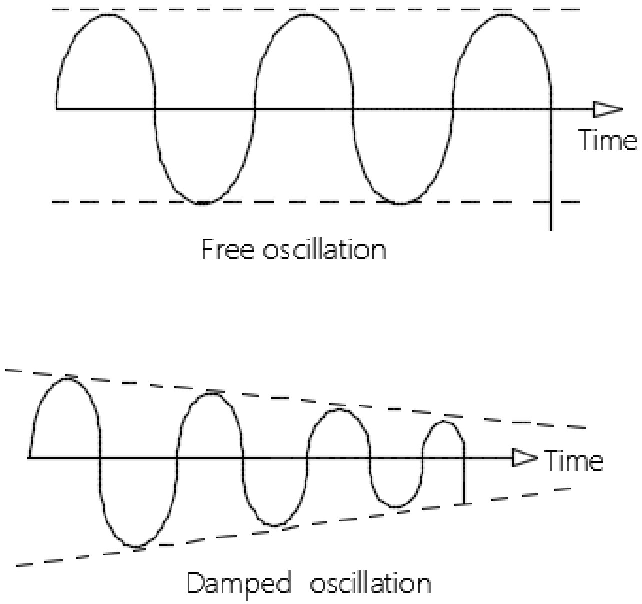
Free Oscillation and Damped Oscillation











Free and damped oscillations

Reference

- https://courses.lumenlearning.com/suny-osuniversityphysics/chapter/15-5-damped-oscillations/
- https://www.google.com/search?q=damped+oscillation+in+pedulum&tbm=isch&ved=2ahUKEwib4_vDsqzpAhUSA94KHcPxBe4Q2-cCegQIABAA&oq=damped+oscillation+in+pedulum&gs_lcp=CgNpbWcQAzoECAAQEzoICAAQCBAeEBNQpiZYq1xggWNoAXAAeACAAaQDiAGsKpIBCDItMTEuNi4ymAEAoAEBqgELZ3dzLXdpei1pbWc&sclient=img&ei=V5O5XtvbEpKG-AbD45fwDg&bih=698&biw=1478&rlz=1C1GGRV_enBD789BD789#imgrc=I87e3Yba5bifcM
- https://www.quora.com/Does-frequency-change-in-damped-vibrations

Differential equation of a damped oscillator

If damping is taken into consideration for an oscillator, then oscillator experiences

(i) Restoring Force : $F_r = -ky$; k=force constant

(ii) Damping Force : $F_d = -b\frac{dy}{dt}$; b=damping constant

Where, y is the displacement of oscillating system and v is the velocity of this displacement.

We, therefore, can write the equation of the damped harmonic oscillator as, $F = F_d + F_r$

From Newton's 2nd law of motion, $F = m\frac{d^2y}{dt^2}$

Combination of Hook's law and Newton's 2nd law of motion:

$$m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt}$$

$$\Rightarrow \frac{d^2y}{dt^2} + \frac{k}{m}y + \frac{b}{m}\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} + 2p\frac{dy}{dt} + \omega^2 y = 0 \qquad (4.1)$$

$2p = \frac{b}{m}$ = damping co-efficient of the medium.

p has the dimension of frequency referred to as damping frequency.

Solution:

To solve equation (4.1) let us take the trial solution,

$$y = Ae^{m't} \qquad (4.2)$$

Substituting this solution in equation (4.1) we get,

$$m'^2 Ae^{m't} + 2pm'Ae^{m't} + \omega^2 Ae^{m't} = 0$$

$$\Rightarrow m'^2 y + 2pm'y + \omega^2 y = 0$$

$\Rightarrow m'^2 + 2pm' + \omega^2 = 0$; [Quadratic equation]

Solving this equation for m' we get,

$$m' = -\frac{2p \pm \sqrt{4p^2 - 4\omega^2}}{2} = -p \pm \sqrt{p^2 - \omega^2}$$

Various Conditions of Damped Oscillation

Then, the general solution of equation (4.1) is,

$$y = e^{-pt}\left[Ae^{\left(\sqrt{p^2-\omega^2}\right)t} + Be^{-\left(\sqrt{p^2-\omega^2}\right)t}\right] \qquad (4.3)$$

Case. I (Overdamped motion)

If $p^2 > \omega^2$,the indices of *"e"* are real and we get,

$$y = e^{-pt}[Ae^{\alpha t} + Be^{-\alpha t}] \qquad (4.4)$$

Where, $\alpha = \sqrt{p^2 - \omega^2}$

Now, let us replace A and B by two other constants C and δ such that we can write, $A=\frac{C}{2}e^{\delta}$ and $B=\frac{C}{2}e^{-\delta}$

Here, $A+B=\frac{C}{2}e^{\delta} + \frac{C}{2}e^{-\delta} = \frac{C}{2}\left(e^{\delta} + e^{-\delta}\right) = \frac{C}{2}2cosh\,\delta$

$\therefore A + B = Ccosh\delta$

$$\frac{A}{B} = \frac{\frac{C}{2}e^{\delta}}{\frac{C}{2}e^{-\delta}} = e^{2\delta}$$

Using the new constants in equation (4.4),

$$y = e^{-pt}\left[\frac{C}{2}e^{\delta}e^{\alpha t} + \frac{C}{2}e^{-\delta}e^{-\alpha t}\right]$$

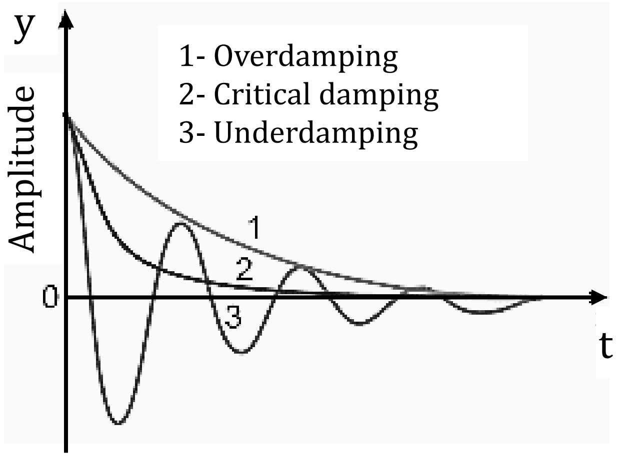
$$= \frac{C}{2}e^{-pt}\left[e^{(\alpha t+\delta)} + e^{-(\alpha t+\delta)}\right]$$

$$= \frac{C}{2}e^{-pt} \times 2\,cosh(\alpha t + \delta)$$

$$= Ce^{-pt}cosh(\alpha t + \delta)$$

So, $y=Ce^{-pt}cosh\left[\left(\sqrt{p^2-\omega^2}\,t\right) + \delta\right] \qquad (4.5)$

Negative power of *"e"* indicates exponential decrease of y that means the particle does not oscillate. Equation (4.5) represents a continuous return of y from its maximum value to zero at t=∞ without oscillation. This type of motion is called the overdamped or dead beat or aperiodic motion.



Example:

Dead beat galvanometer, pendulum oscillating in a viscous fluid, etc.

Then, the general solution of equation (4.1) is,

$$y = e^{-pt}\left[Ae^{\left(\sqrt{p^2-\omega^2}\right)t} + Be^{-\left(\sqrt{p^2-\omega^2}\right)t}\right] \tag{4.3}$$

Case. II (Underdamped motion)

If $p^2<\omega^2$,the indices of *"e"* are imaginary and we get,

Where, $\theta=\sqrt{(\omega^2 - p^2)}$

$$y = e^{-pt}\left[Ae^{i\theta t} + Be^{-i\theta t}\right]$$

$$=e^{-pt}[A cos\theta t + iA sin\theta t + B cos\theta t - iB sin\theta t]$$

$$=e^{-pt}[(A + B)cos\theta t + i(A - B)sin\theta t] \tag{4.5}$$

Let, $(A+B)=a\,cos\gamma$ and $i(A\text{-}B)=a\,sin\gamma$

$$a = \sqrt{a^2cos^2\gamma + a^2sin^2\gamma} = \sqrt{(A + B)^2 + i^2(A - B)^2}$$

$$=\sqrt{A^2 + 2AB + B^2 - A^2 + 2AB - B^2} = \pm 2\sqrt{AB}$$

$$\tan\gamma=\frac{asin\gamma}{acos\gamma}=\frac{i(A-B)}{(A+B)}$$

Using the new constants in equation (4.5),

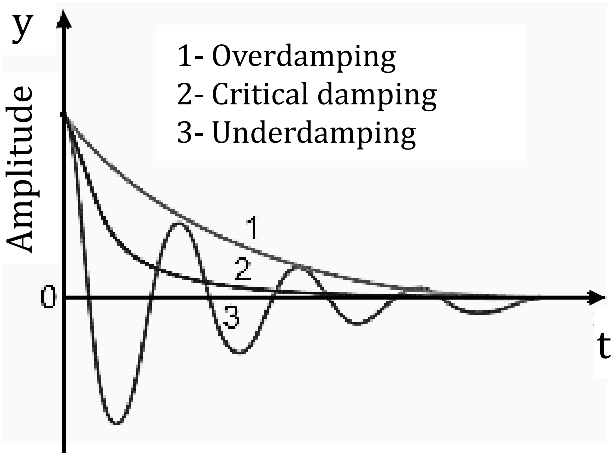
$$y = e^{-pt}[acos\gamma\, cos\theta t + asin\gamma\, sin\theta t]$$

$$y=ae^{-pt}[cos\theta t\, cos\gamma + sin\theta t\, sin\gamma]$$

$$=ae^{-pt}cos(\theta t-\gamma)$$

$$y=ae^{-pt}cos\left[\sqrt{(\omega^2 - p^2)}t-\gamma\right] \tag{4.6}$$

In this case y alternates in sign and we have periodic motion but the amplitude continuously diminishes due to the factor e^{-pt}. This situation is called underdamping with the amplitude ae^{-pt} and the frequency $\sqrt{(\omega^2 - p^2)}$.



Then, the general solution of equation (4.1) is,

$$y = e^{-pt}\left[Ae^{\left(\sqrt{p^2-\omega^2}\right)t} + Be^{-\left(\sqrt{p^2-\omega^2}\right)t}\right] \qquad (4.3)$$

Case. III (Critical damping motion)

If $p^2=\omega^2$, $(p^2-\omega^2)=0$; So, $p^2=\omega^2, p=\omega$

From equation (4.3) we can write,

$$y = e^{-\omega t}[Ae^0 + Be^0]$$

$$=e^{-\omega t}[A+B]$$

It implies that the oscillation is decaying without any damping factor. It is not possible. So, the solution breaks down. Now, we have to consider that p^2 is not quite equal to ω^2, but very close to each other. Thus $\sqrt{p^2-\omega^2} = h \approx 0$ (close to zero but not zero).

From equation (Using the new constants in equation (4.3),

$$y = e^{-pt}[Ae^{ht} + Be^{-ht}] = e^{-pt}\left[A\left(1+ht+\frac{h^2t^2}{2!}+\frac{h^3t^3}{3!}+\cdots\right) + B\left(1-ht+\frac{h^2t^2}{2!}-\frac{h^3t^3}{3!}+\cdots\right)\right] = e^{-pt}[A(1+ht)] + B(1-ht)]$$

$$y=e^{-pt}[(A+B)+(A-B)ht] \qquad (4.7)$$

Let, $A+B=A'$ and $(A-B)h=B'$

$$y=e^{-pt}[A'+B't] \qquad (4.8)$$

At amplitude, $y=y_{max}=a$ (at $t=0$)

Applying these two conditions in equation (4.8),

$$a=e^0(A'+B'\times 0) \Rightarrow A'=a$$

$$\frac{dy}{dt} = -pe^{-pt}(A'+B't) + e^{-pt}B'$$

$$\left[\frac{dy}{dt}\right]_{t=0} = -pe^0(A'+B'\times 0) + e^0B' = 0$$

$$\Rightarrow -pA' + B' = 0$$

$$\Rightarrow B' = pa$$

So, from equation (4.8)

$$y = e^{-pt}[a+pat]$$

$$y = ae^{-pt}[1+pt] \qquad (4.9)$$

This solution represents a continuous return of y from its amplitude to zero. Although it looks like overdamped motion it is actually a boundary between underdamped and overdamped motion. Under this condition oscillatory motion changes over to dead beat motion and vice versa. Hence, this is called critical damping motion.

The Logarithmic Decrement

In the case of an underdamped motion the amplitude of the motion reduces with time following a particular fashion. Let us calculate the decrement of the successive amplitudes at the intervals of time $t=\frac{T}{2}=\frac{\pi}{\omega}$. Let the magnitudes of successive amplitudes be A_1, A_2, A_3, A_4, etc. Using the expression of amplitude ae^{-pt} we get,

At time t=0, $A_1=ae^0=a$

At time $t=\frac{T}{2}=\frac{\pi}{\omega}$, $A_2=ae^{-\frac{pT}{2}}$

At time $t=T=\frac{2\pi}{\omega}$, $A_3=ae^{-pT}$

At time $t=\frac{3T}{2}=\frac{3\pi}{\omega}$, $A_4=ae^{-\frac{3pT}{2}}$

$\therefore \frac{A_1}{A_2}=\frac{A_2}{A_3}=\frac{A_3}{A_4}=\ldots\ldots\ldots\ldots=e^{\frac{pT}{2}}=\text{constant}$

Since, p and T are constants for a given motion.

Putting, $\frac{pT}{2}=\lambda$ we have

$\frac{A_1}{A_2}=\frac{A_2}{A_3}=\frac{A_3}{A_4}=\ldots\ldots\ldots\ldots=e^{\lambda}$

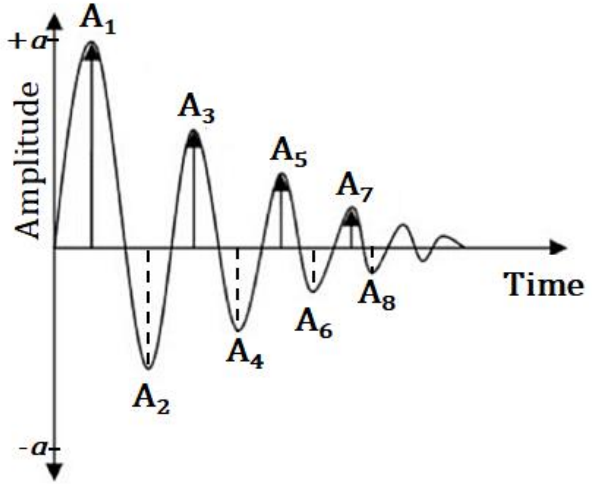
$\frac{A_1}{A_2}\times\frac{A_2}{A_3}\times\frac{A_3}{A_4}\times \ldots\ldots\ldots \frac{A_{n-1}}{An}\times\frac{A_n}{An+1} = e^{\lambda}\times e^{\lambda}\times e^{\lambda}\times\ldots\ldots\ldots e^{\lambda}$up to nth term ; Here, n=1, 2, 3, ………

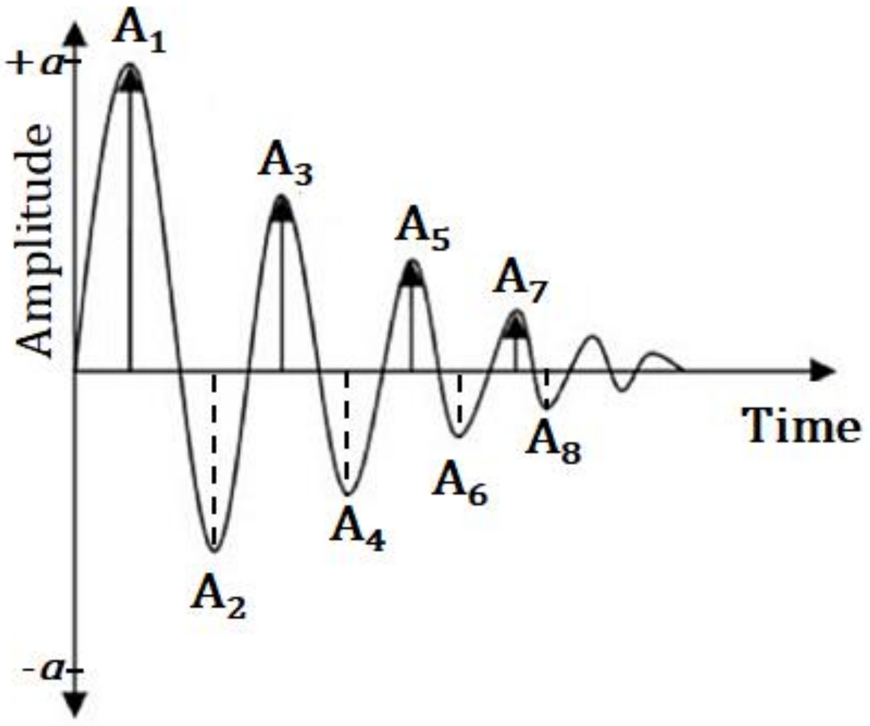
$\therefore \frac{A_1}{An+1}=e^{\lambda+\lambda+\lambda+\text{.........up to nth term}} \Rightarrow \frac{A_1}{An+1}=e^{n\lambda}$

$\Rightarrow log_e\frac{A_1}{An+1}=n\lambda$

$$\therefore \lambda=\frac{1}{n}log_e\frac{A_1}{An+1} \qquad (4.10)$$

λ in equation (4.10) is called the logarithmic decrement.





- Angular frequency of a damped oscillator, $\omega' = \sqrt{\omega^2 - p^2}$

- Since, $\omega^2 = \frac{k}{m}$ and $2p = \frac{b}{m}$; $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

- Mechanical energy of a free oscillator, $E = \frac{1}{2}ka^2 =$constant

- Mechanical energy of a damped oscillator, $E = \frac{1}{2}ka^2 e^{-2pt} = \frac{1}{2}ka^2 e^{-\frac{b}{m}t}$; [reduces with exponentially with time]

Fundamentals of Physics – David Halliday, Robert Resnick, and Jearls Walker (6th Ed.), Chapter:16, page no. 361

Sample Problem 16-7

For the damped oscillator of Fig. 16-15, $m = 250$ g, $k = 85$ N/m, and $b = 70$ g/s.

(a) What is the period of the motion?

SOLUTION: The **Key Idea** here is that because $b \ll \sqrt{km} = 4.6$ kg/s, the period is approximately that of the undamped oscillator. From Eq. 16-13, we then have

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.25\text{ kg}}{85\text{ N/m}}} = 0.34\text{ s.} \quad \text{(Answer)}$$

(b) How long does it take for the amplitude of the damped oscillations to drop to half its initial value?

SOLUTION: Now the **Key Idea** is that the amplitude at time t is displayed in Eq. 16-40 as $x_m e^{-bt/2m}$. It has the value x_m at $t = 0$. Thus, we must find the value of t for which

$$x_m e^{-bt/2m} = \tfrac{1}{2}x_m.$$

Canceling x_m and taking the natural logarithm of the equation that remains, we have $\ln\frac{1}{2}$ on the right side and

$$\ln(e^{-bt/2m}) = -bt/2m$$

on the left side. Thus,

$$t = \frac{-2m\ln\frac{1}{2}}{b} = \frac{-(2)(0.25\text{ kg})(\ln\frac{1}{2})}{0.070\text{ kg/s}}$$

$$= 5.0\text{ s.} \quad \text{(Answer)}$$

Because $T = 0.34$ s, this is about 15 periods of oscillation.

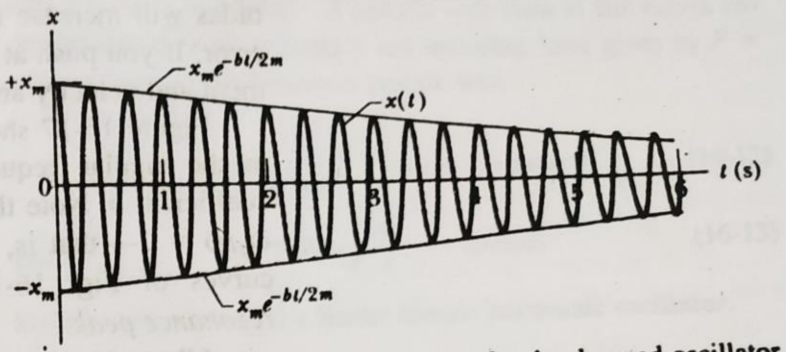


Fig. 16-16 The displacement function $x(t)$ for the damped oscillator of Fig. 16-15, with values given in Sample Problem 16-7. The amplitude, which is $x_m e^{-bt/2m}$, decreases exponentially with time.

(c) How long does it take for the mechanical energy to drop to one-half its initial value?

SOLUTION: Here the **Key Idea** is that, from Eq. 16-42, the mechanical energy at time t is $\frac{1}{2}kx_m^2 e^{-bt/m}$. It has the value $\frac{1}{2}kx_m^2$ at $t = 0$. Thus, we must find the value of t for which

$$\tfrac{1}{2}kx_m^2 e^{-bt/m} = \tfrac{1}{2}(\tfrac{1}{2}kx_m^2).$$

If we divide both sides of this equation by $\frac{1}{2}kx_m^2$ and solve for t as we did above, we find

$$t = \frac{-m\ln\frac{1}{2}}{b} = \frac{-(0.25\text{ kg})(\ln\frac{1}{2})}{0.070\text{ kg/s}} = 2.5\text{ s.} \quad \text{(Answer)}$$

This is exactly half the time we calculated in (b), or about 7.5 periods of oscillation. Figure 16-16 was drawn to illustrate this sample problem.

Solution:

b= 0.07kg/s , $\sqrt{km}=\sqrt{(85\text{N/m})(0.25\text{ kg})}$ =4.6 kg/s; Thus, b<<$\sqrt{km}$ (~66 times less)

(a) $T=\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.25\ kg}{85\ M/m}} = 0.34$ s

(b) $A = ae^{-pt} = ae^{-\frac{bt}{2m}}$

Now, $ae^{-\frac{bt}{2m}} = \frac{a}{2} \Rightarrow e^{-\frac{bt}{2m}} = \frac{1}{2}$

$\Rightarrow log_e e^{-\frac{bt}{2m}} = log_e \frac{1}{2}$

$\Rightarrow -\frac{bt}{2m} = log_e \frac{1}{2}$

So, $t = \frac{-2m\,log_e\frac{1}{2}}{b} = \frac{-(2)(0.25\ kg)\,log_e\frac{1}{2}}{0.070\ kg/s} = 5.0$ s

In terms of T: (5.0/0.34)=14.75 T≈15 T

(c) $E = \frac{1}{2}ka^2 e^{-\frac{bt}{m}}$

$\frac{1}{2}ka^2 e^{-\frac{bt}{m}} = \frac{1}{2}(\frac{1}{2}ka^2) \Rightarrow e^{-\frac{bt}{m}} = \frac{1}{2}$

$t = \frac{-m\,log_e\frac{1}{2}}{b} = \frac{-(0.25\ kg)\,log_e\frac{1}{2}}{0.070\ kg/s} = 2.5$ s

In terms of T: (2.5/0.34)=7.35 T≈7.5 T