Waves and Oscillations

Lecture No. 1

Topics: Simple Harmonic Motion and related systems

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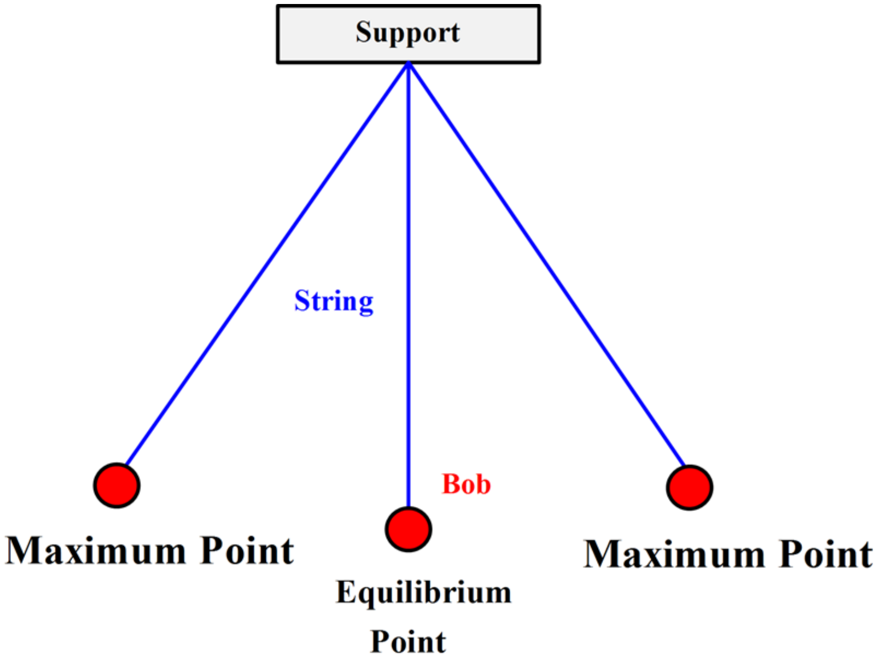
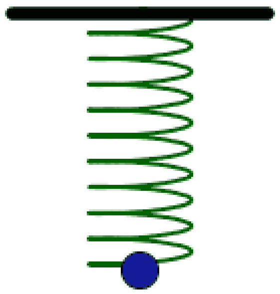
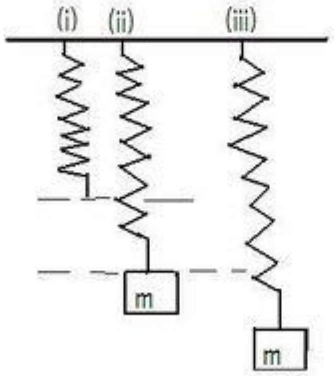
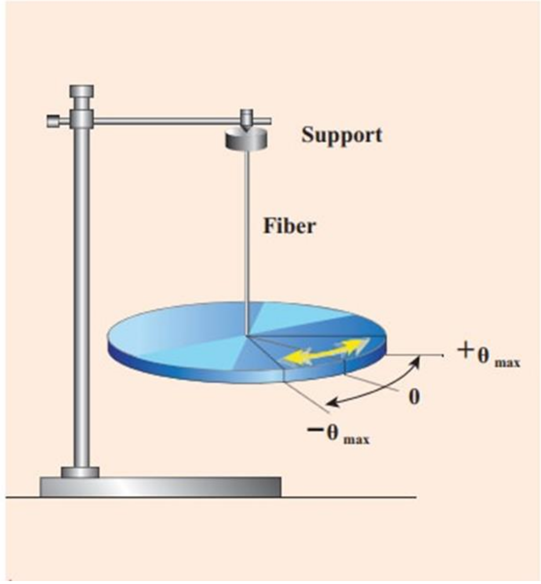
Characteristics of simple harmonic motion (SHM)

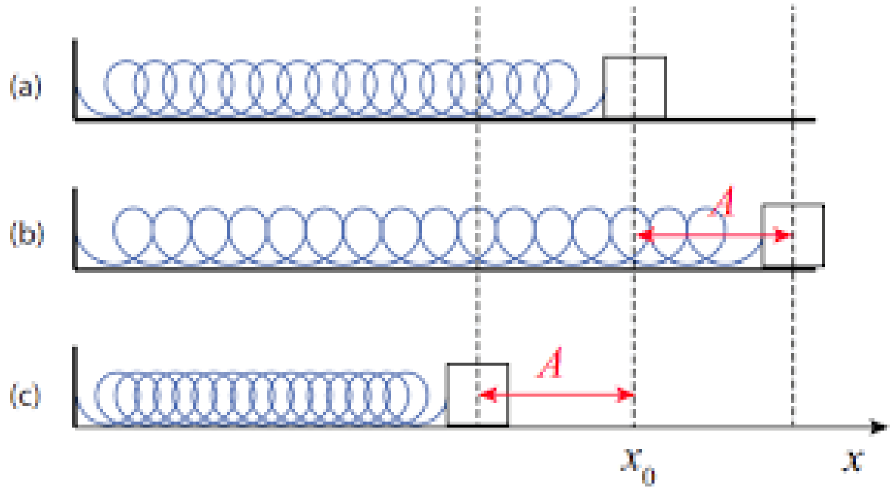
- It is a periodic motion.
- The acceleration of the particle (force on the particle) is directly proportional to its displacement.
- Force on the particle (acceleration of the particle) is directed towards its equilibrium position.
- The total energy of the particle executing SHM is conserved.
- The maximum displacement of the particle on either sides of the equilibrium position is the same.


Support
String
Bob
Maximum Point
Equilibrium Point
Maximum Point




(i)
(ii)
(iii)
m
m


Support
Fiber
$+\theta_{max}$
0
$-\theta_{max}$


(a)
(b)
(c)
A
A
x_0
x

Restoring Force

- Restoring force is a force which acts to bring a body to its equilibrium position.
- It is a function only of position of the particle.
- It is always directed back toward the equilibrium position of the system.

Hooke's law

- If a force (F) needed to extend or compress a spring or to displace a body by some distance (y) from its equilibrium position, then F is proportional to y. So, $F \propto -y$ or, is, $F = -ky$. (-ve sign indicates that F is directed opposite to y)
- Here k is a constant factor, known as the force constant. When $y=1$, $F=-k$
- k is defined as the amount of restoring force required to produce unit displacement.

Differential equation of a simple harmonic oscillator

Hooke's law, $F = -ky$

F=restoring force, k=force constant

Newton's 2nd law of motion,

$F = mass \times acceleration = m\frac{d^2y}{dt^2}$

Combination of Hooke's law and Newton's 2nd law of motion:

$F = -ky = m\frac{d^2y}{dt^2}$

$\Rightarrow m\frac{d^2y}{dt^2} + ky = 0$

$\Rightarrow \frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \qquad (1.1) \qquad \omega = \sqrt{\frac{k}{m}}$ = angular frequency

$\therefore \frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad (1.2)$

Equation (1.2) is the differential equation of a simple harmonic oscillator.

Solution:

Rewriting equation (1.2), $\frac{d^2y}{dt^2} = -\omega^2 y$

1st order derivative

2nd order derivative

$\Rightarrow \int 2\left(\frac{dy}{dt}\right)\frac{d^2y}{dt^2}\,dt = -2\omega^2 \int \left(\frac{dy}{dt}\right) y\,dt$

0th order derivative

$\Rightarrow \left(\frac{dy}{dt}\right)^2 = -\omega^2 y^2 + C$

$\Rightarrow \left(\frac{dy}{dt}\right)^2 = -\omega^2 y^2 + C \qquad (1.3)$

Here, C=constant for integration. Now, applying boundary conditions, at $y_{max} = a$, $\frac{dy}{dt} = 0$ [since, kinetic energy is zero at maximum displacement].

From equation (1.3) we get, $0 = -\omega^2 a^2 + C$

So, $C = \omega^2 a^2$

Now, $\left(\frac{dy}{dt}\right)^2 = \omega^2 (a^2 - y^2)$

$\Rightarrow \frac{dy}{dt} = \pm\omega\sqrt{(a^2 - y^2)}$

$\Rightarrow \int \frac{dy}{\sqrt{(a^2 - y^2)}} = \omega \int dt$

$\Rightarrow \sin^{-1}\frac{y}{a} = \omega t + \varphi$

$\therefore y = a\sin(\omega t + \varphi) \qquad (1.4)$

Various Equations Related to SHM

Equations of Displacement

- $y = a\sin(\omega t + \varphi)$ (1.4)
- $y = a\sin\omega t\cos\varphi + a\cos\omega t\sin\varphi$ (1.5)
- $y = A\sin\omega t + B\cos\omega t$ (1.6)

Assuming,
$a\cos\varphi = A$
$a\sin\varphi = B$

- Velocity, $\frac{dy}{dt} = \frac{d}{dt}[a\sin(\omega t + \varphi)] = \omega a\cos(\omega t + \varphi)$

Or, $\frac{dy}{dt} = \omega a\sqrt{1 - \sin^2(\omega t + \varphi))}$

Or, $\frac{dy}{dt} = \omega\sqrt{a^2 - a^2\sin^2(\omega t + \varphi))}$

Or, $\frac{dy}{dt} = \omega\sqrt{(a^2 - y^2)}$ (1.7)

- Maximum velocity, $\left(\frac{dy}{dt}\right)_{\max} = \omega a$ (1.8) [When, $y=0$]
- Minimum velocity, $\left(\frac{dy}{dt}\right)_{\min} = 0$ [When, $y=a$]

- Acceleration, $\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}[\omega a\cos(\omega t + \varphi)]$

Or, $\frac{d^2y}{dt^2} = -\omega^2 a\sin(\omega t + \varphi)$

So, $\frac{d^2y}{dt^2} = -\omega^2 y$ (1.9)

- Maximum Acceleration

$\left(\frac{d^2y}{dt^2}\right)_{\max} = -\omega^2 a$ (1.10) [When, $y=a$]

- Minimum Acceleration

$\left(\frac{d^2y}{dt^2}\right)_{\min} = 0$ [When, $y=0$]

Various Equations Related to SHM

Time period

The equation of particle displacement at the time t is,

$$y = a\sin(\omega t + \varphi)$$

The equation of particle displacement at the time $(t+\frac{2\pi}{\omega})$ is,

$$y = a\sin[\omega(t+\frac{2\pi}{\omega}) + \varphi)$$

$$= a\sin(\omega t+2\pi + \varphi)$$

$$= a\sin(\omega t + \varphi)$$

The equation of particle displacement at the time $(t+\frac{2\pi}{\omega})$ is the same as that at time t. So, it can be said that the motion is repeated after every $\frac{2\pi}{\omega}$ seconds.

Time period, $T = \frac{2\pi}{\omega}$

Since, Angular frequency, $\omega=\sqrt{\frac{k}{m}}$

$$T=2\pi\sqrt{\frac{m}{k}} \qquad (1.10)$$

- <u>Initial phase, φ</u>
- If at t=0, y=0, then φ=0; (counting is started at the equilibrium) $y = a\sin\omega t$
- If t=0, y=a, then φ=π/2; (counting is started at the amplitude) y= $a\sin(\omega t + \pi/2) = a\cos\omega t$
- If t=t′, y=0, (counting is started before the equ. position) then, y= $a\sin(\omega t - e)$
- If t=t′, y=0, (counting is started after the equ. position) then, y= $a\sin(\omega t+e)$