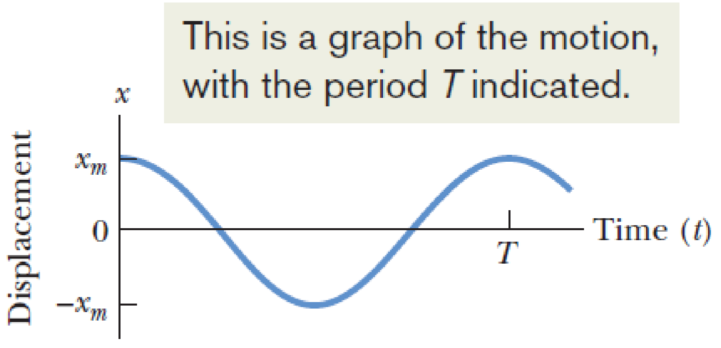
Waves and Oscillations

Topic: Sample problems on Simple Harmonic Motion

Teacher's name: Dr. Mehnaz Sharmin



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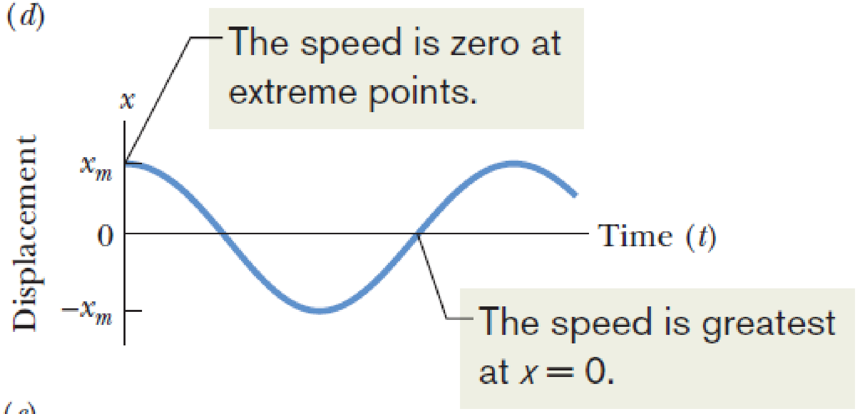

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Checkpoint 1

A particle undergoing simple harmonic oscillation of period T (like that in Fig. 15-2) is at $-x_m$ at time $t = 0$. Is it at $-x_m$, at $+x_m$, at 0, between $-x_m$ and 0, or between 0 and $+x_m$ when (a) $t = 2.00T$, (b) $t = 3.50T$, and (c) $t = 5.25T$?

$x = x_m cos(\omega t + \varphi)$ is the equation of displacement

Checkpoint1:

Since, counting is started at the amplitude, that is at $t=0, x=-x_m$

So, $-x_m = x_m cos(0 + \varphi) \Rightarrow cos\varphi = -1 \Rightarrow \varphi = cos^{-1}(-1) = \pi$

The equation of displacement is, $\boldsymbol{x = x_m cos(\omega t - \pi)}$

The position of the particle at the following times are to be determined.

(a) $t=2.0T=2\frac{2\pi}{\omega}=\frac{4\pi}{\omega}$; $x = x_m cos(\omega\frac{4\pi}{\omega} - \pi) = x_m cos(4\pi - \pi) = xm\,cos\,3\pi = x_m(-1) = -x_m$

(b) $t=3.50T=3.50\frac{2\pi}{\omega}=\frac{7\pi}{\omega}$; $x = x_m cos(\omega\frac{7\pi}{\omega} - \pi) = x_m cos(7\pi - \pi) = xm\,cos\,6\pi = x_m(+1) = +x_m$

(c) $t=5.25T=5.25\,\frac{2\pi}{\omega}=\frac{10.50\pi}{\omega}$; $x = x_m cos(\omega\frac{10.50\pi}{\omega} - \pi) = x_m cos(10.50\pi - \pi) = xm\,cos\,9.5\pi = x_m \times (0) = 0$

Ref: Fundamentals of Physics – David Halliday, Robert Resnick, and Jearls Walker (10th Ed.), Chapter:15

Checkpoint 2

Which of the following relationships between a particle's acceleration a and its position x indicates simple harmonic oscillation: (a) $a = 3x^2$, (b) $a = 5x$, (c) $a = -4x$, (d) $a = -2/x$? For the SHM, what is the angular frequency (assume the unit of rad/s)?

Checkpoint 3

Which of the following relationships between the force F on a particle and the particle's position x gives SHM: (a) $F = -5x$, (b) $F = -400x^2$, (c) $F = 10x$, (d) $F = 3x^2$?

CHECKPOINT 2: Which of the following relationships between the force F on a particle and the particle's position x implies simple harmonic oscillation: (a) $F = -5x$, (b) $F = -400x^2$, (c) $F = 10x$, (d) $F = 3x^2$?

The equation must satisfy Hooke's law, F=-kx

(a) $F = -5x; \Rightarrow$ SHM

Because this equation shows that displacement is opposite to the restoring force.

(b) $F = -400x^2; \Rightarrow$ not a SHM

Although there is a negative sign, it contains square of *x.*

(c) $F = 10x; \Rightarrow$ not a SHM

Because this equation shows that displacement is not opposite to the restoring force.

(d) $F = 3x^2; \Rightarrow$ not a SHM

Because this equation shows that displacement is not opposite to the restoring force and it contains square of x.

Sample Problem 15.01 Block–spring SHM, amplitude, acceleration, phase constant

A block whose mass m is 680 g is fastened to a spring whose spring constant k is 65 N/m. The block is pulled a distance $x = 11$ cm from its equilibrium position at $x = 0$ on a frictionless surface and released from rest at $t = 0$.

(a) What are the angular frequency, the frequency, and the period of the resulting motion?

KEY IDEA

The block–spring system forms a linear simple harmonic oscillator, with the block undergoing SHM.

Calculations: The angular frequency is given by Eq. 15-12:

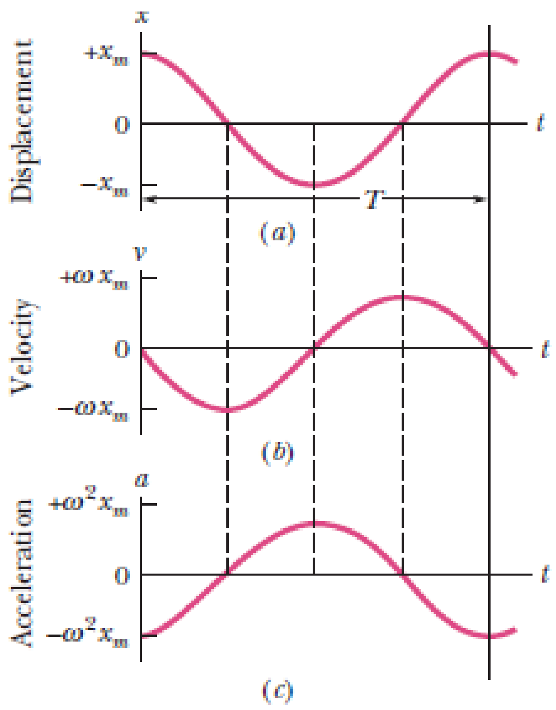
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{65\ \text{N/m}}{0.68\ \text{kg}}} = 9.78\ \text{rad/s}$$
$$\approx 9.8\ \text{rad/s.} \qquad \text{(Answer)}$$

The frequency follows from Eq. 15-5, which yields

$$f = \frac{\omega}{2\pi} = \frac{9.78\ \text{rad/s}}{2\pi\ \text{rad}} = 1.56\ \text{Hz} \approx 1.6\ \text{Hz.} \qquad \text{(Answer)}$$

The period follows from Eq. 15-2, which yields

$$T = \frac{1}{f} = \frac{1}{1.56\ \text{Hz}} = 0.64\ \text{s} = 640\ \text{ms.} \qquad \text{(Answer)}$$



(b) What is the amplitude of the oscillation?

KEY IDEA

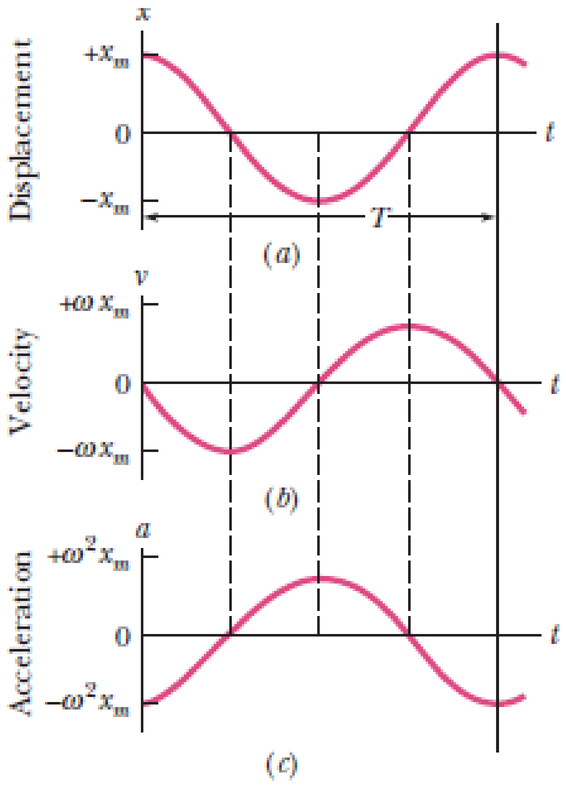
With no friction involved, the mechanical energy of the spring–block system is conserved.

Reasoning: The block is released from rest 11 cm from its equilibrium position, with zero kinetic energy and the elastic potential energy of the system at a maximum. Thus, the block will have zero kinetic energy whenever it is again 11 cm from its equilibrium position, which means it will never be farther than 11 cm from that position. Its maximum displacement is 11 cm:

$$x_m = 11\ \text{cm.} \qquad \text{(Answer)}$$

This maximum speed occurs when the oscillating block is rushing through the origin; compare Figs. 15-6*a* and 15-6*b*, where you can see that the speed is a maximum whenever $x = 0$.

(d) What is the magnitude a_m of the maximum acceleration of the block?



(c) What is the maximum speed v_m of the oscillating block, and where is the block when it has this speed?

KEY IDEA

The maximum speed v_m is the velocity amplitude ωx_m in Eq. 15-6.

Calculation: Thus, we have

$$v_m = \omega x_m = (9.78 \text{ rad/s})(0.11 \text{ m})$$
$$= 1.1 \text{ m/s.} \qquad \text{(Answer)}$$

This maximum speed occurs when the oscillating block is rushing through the origin; compare Figs. 15-6*a* and 15-6*b*, where you can see that the speed is a maximum whenever $x = 0$.

(d) What is the magnitude a_m of the maximum acceleration of the block?

KEY IDEA

The magnitude a_m of the maximum acceleration is the acceleration amplitude $\omega^2 x_m$ in Eq. 15-7.

Calculation: So, we have

$$a_m = \omega^2 x_m = (9.78 \text{ rad/s})^2(0.11 \text{ m})$$
$$= 11 \text{ m/s}^2. \qquad \text{(Answer)}$$

This maximum acceleration occurs when the block is at the ends of its path, where the block has been slowed to a stop so that its motion can be reversed. At those extreme points, the force acting on the block has its maximum magnitude; compare Figs. 15-6*a* and 15-6*c*, where you can see that the magnitudes of the displacement and acceleration are maximum at the same times, when the speed is zero, as you can see in Fig. 15-6*b*.

$$x(t) = x_m \cos(\omega t + \phi) \quad \text{(displacement),} \qquad (15\text{-}3)$$

(e) What is the phase constant ϕ for the motion?

Calculations: Equation 15-3 gives the displacement of the block as a function of time. We know that at time $t = 0$, the block is located at $x = x_m$. Substituting these *initial conditions*, as they are called, into Eq. 15-3 and canceling x_m give us

$$1 = \cos \phi. \qquad (15\text{-}14)$$

Taking the inverse cosine then yields

$$\phi = 0 \text{ rad.} \qquad \text{(Answer)}$$

(Any angle that is an integer multiple of 2π rad also satisfies Eq. 15-14; we chose the smallest angle.)

(f) What is the displacement function $x(t)$ for the spring–block system?

Calculation: The function $x(t)$ is given in general form by Eq. 15-3. Substituting known quantities into that equation gives us

$$x(t) = x_m \cos(\omega t + \phi)$$
$$= (0.11 \text{ m}) \cos[(9.8 \text{ rad/s})t + 0]$$
$$= 0.11 \cos(9.8t), \qquad \text{(Answer)}$$

Sample Problem 15.02 Finding SHM phase constant from displacement and velocity

At $t = 0$, the displacement $x(0)$ of the block in a linear oscillator like that of Fig. 15-7 is -8.50 cm. (Read $x(0)$ as "x at time zero.") The block's velocity $v(0)$ then is -0.920 m/s, and its acceleration $a(0)$ is $+47.0$ m/s^2.

(a) What is the angular frequency ω of this system?

KEY IDEA

With the block in SHM, Eqs. 15-3, 15-6, and 15-7 give its displacement, velocity, and acceleration, respectively, and each contains ω.

Calculations: Let's substitute $t = 0$ into each to see whether we can solve any one of them for ω. We find

$$x(0) = x_m \cos\phi, \qquad (15\text{-}15)$$

$$v(0) = -\omega x_m \sin\phi, \qquad (15\text{-}16)$$

and

$$a(0) = -\omega^2 x_m \cos\phi. \qquad (15\text{-}17)$$

In Eq. 15-15, ω has disappeared. In Eqs. 15-16 and 15-17, we know values for the left sides, but we do not know x_m and ϕ. However, if we divide Eq. 15-17 by Eq. 15-15, we neatly eliminate both x_m and ϕ and can then solve for ω as

$$\omega = \sqrt{-\frac{a(0)}{x(0)}} = \sqrt{-\frac{47.0\ \text{m/s}^2}{-0.0850\ \text{m}}}$$
$$= 23.5\ \text{rad/s}. \qquad \text{(Answer)}$$

(b) What are the phase constant ϕ and amplitude x_m?

Calculations: We know ω and want ϕ and x_m. If we divide Eq. 15-16 by Eq. 15-15, we eliminate one of those unknowns and reduce the other to a single trig function:

$$\frac{v(0)}{x(0)} = \frac{-\omega x_m \sin\phi}{x_m \cos\phi} = -\omega \tan\phi.$$

Solving for $\tan\phi$, we find

$$\tan\phi = -\frac{v(0)}{\omega x(0)} = -\frac{-0.920\ \text{m/s}}{(23.5\ \text{rad/s})(-0.0850\ \text{m})}$$
$$= -0.461.$$

This equation has two solutions:

$$\phi = -25^\circ \quad \text{and} \quad \phi = 180^\circ + (-25^\circ) = 155^\circ.$$

Normally only the first solution here is displayed by a calculator, but it may not be the physically possible solution. To choose the proper solution, we test them both by using them to compute values for the amplitude x_m. From Eq. 15-15, we find that if $\phi = -25^\circ$, then

$$x_m = \frac{x(0)}{\cos\phi} = \frac{-0.0850\ \text{m}}{\cos(-25^\circ)} = -0.094\ \text{m}.$$

We find similarly that if $\phi = 155^\circ$, then $x_m = 0.094$ m. Because the amplitude of SHM must be a positive constant, the correct phase constant and amplitude here are

$$\phi = 155^\circ \quad \text{and} \quad x_m = 0.094\ \text{m} = 9.4\ \text{cm}. \qquad \text{(Answer)}$$

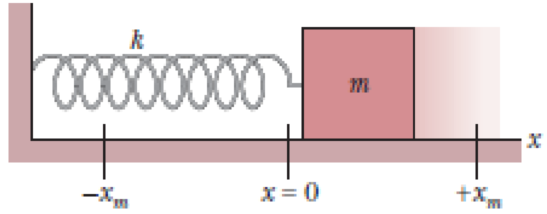


Figure 15-7 A linear simple harmonic oscillator. The surface is frictionless. Like the particle of Fig. 15-2, the block moves in simple harmonic motion once it has been either pulled or pushed away from the $x = 0$ position and released. Its displacement is then given by Eq. 15-3.

$$x(t) = x_m \cos(\omega t + \phi) \quad \text{(displacement)},$$

$$v(t) = -\omega x_m \sin(\omega t + \phi) \quad \text{(velocity)},$$

$$a(t) = -\omega^2 x_m \cos(\omega t + \phi) \quad \text{(acceleration)}.$$

Checkpoint 4

In Fig. 15-7, the block has a kinetic energy of 3 J and the spring has an elastic potential energy of 2 J when the block is at $x = +2.0$ cm. (a) What is the kinetic energy when the block is at $x = 0$? What is the elastic potential energy when the block is at (b) $x = -2.0$ cm and (c) $x = -x_m$?

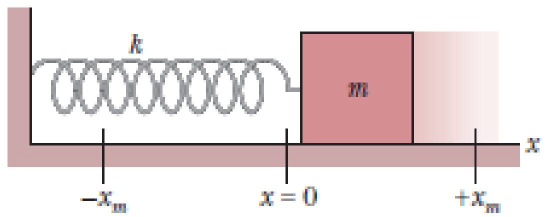


Figure 15-7 A linear simple harmonic oscillator. The surface is frictionless. Like the particle of Fig. 15-2, the block moves in simple harmonic motion once it has been either pulled or pushed away from the $x = 0$ position and released. Its displacement is then given by Eq. 15-3.

$x = x_m cos(\omega t + \varphi)$

$K=\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}kx_m{}^2sin^2(\omega t + \varphi)=\frac{1}{2}kx_m{}^2[1\text{-}cos^2(\omega t + \varphi)] =\frac{1}{2}k\ [x_m{}^2\text{-}x^2]$

$U=\frac{1}{2}kx_m{}^2cos^2(\omega t + \varphi)=\frac{1}{2}kx^2$

When the block is at *x=2.0 cm*

So, $K = 3J =\frac{1}{2}kx_m{}^2 - \frac{1}{2}k \times (0.02m)^2$ And $U= 2J=\frac{1}{2}k \times (0.02m)^2$

$\Rightarrow \frac{1}{2}kx_m{}^2 = 3J + \frac{1}{2}k \times (0.02m)^2$

Finally, $\frac{1}{2}kx_m{}^2 = 3J+2J=5J$

(a) When the block is at $x=0$, $U=0$, $K=\frac{1}{2}kx_m{}^2 =5J$

(b) When the block is at $x=-2.0$ cm, $U=\frac{1}{2}k(-0.02m)^2= \frac{1}{2}k(0.02m)^2=2J$

(c) When the block is at $x=-x_m$, $U=\frac{1}{2}kx_m{}^2=5J$

Sample Problem 15.03

Suppose the block has mass $m = 2.72 \times 10^5$ kg and is designed to oscillate at frequency $f = 10.0$ Hz and with amplitude $x_m = 20.0$ cm.

(a) What is the total mechanical energy E of the spring–block system?

KEY IDEA

The mechanical energy E (the sum of the kinetic energy $K = \frac{1}{2}mv^2$ of the block and the potential energy $U = \frac{1}{2}kx^2$ of the spring) is constant throughout the motion of the oscillator. Thus, we can evaluate E at any point during the motion.

Calculations: Because we are given amplitude x_m of the oscillations, let's evaluate E when the block is at position $x = x_m$, where it has velocity $v = 0$. However, to evaluate U at that point, we first need to find the spring constant k. From Eq. 15-12 ($\omega = \sqrt{k/m}$) and Eq. 15-5 ($\omega = 2\pi f$), we find

$$k = m\omega^2 = m(2\pi f)^2$$
$$= (2.72 \times 10^5 \text{ kg})(2\pi)^2(10.0 \text{ Hz})^2$$
$$= 1.073 \times 10^9 \text{ N/m}.$$

$$= (2.72 \times 10^5 \text{ kg})(2\pi)^2(10.0 \text{ Hz})^2$$
$$= 1.073 \times 10^9 \text{ N/m}.$$

We can now evaluate E as

$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$
$$= 0 + \tfrac{1}{2}(1.073 \times 10^9 \text{ N/m})(0.20 \text{ m})^2$$
$$= 2.147 \times 10^7 \text{ J} \approx 2.1 \times 10^7 \text{ J}. \qquad \text{(Answer)}$$

(b) What is the block's speed as it passes through the equilibrium point?

Calculations: We want the speed at $x = 0$, where the potential energy is $U = \frac{1}{2}kx^2 = 0$ and the mechanical energy is entirely kinetic energy. So, we can write

$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$
$$2.147 \times 10^7 \text{ J} = \tfrac{1}{2}(2.72 \times 10^5 \text{ kg})v^2 + 0,$$

or $$v = 12.6 \text{ m/s}. \qquad \text{(Answer)}$$

Because E is entirely kinetic energy, this is the maximum speed v_m.

Sample Problem 16-3

(a) What is the mechanical energy E of the linear oscillator of Sample Problem 16-1? (Initially, the block's position is $x = 11$ cm and its speed is $v = 0$. Spring constant k is 65 N/m.)

SOLUTION: The **Key Idea** here is that the mechanical energy E (the sum of the kinetic energy $K = \frac{1}{2}mv^2$ of the block and the potential energy $U = \frac{1}{2}kx^2$ of the spring) is constant throughout the motion of the oscillator. Thus, we can evaluate E at any point during the motion. Because we are given the initial conditions of the oscillator as $x = 11$ cm and $v = 0$, let us evaluate E for those conditions. We find

$$E = K + U = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = 0 + \tfrac{1}{2}(65 \text{ N/m})(0.11 \text{ m})^2$$
$$= 0.393 \text{ J} \approx 0.39 \text{ J}. \qquad \text{(Answer)}$$

(b) What are the potential energy U and kinetic energy K of the oscillator when the block is at $x = \frac{1}{2}x_m$? What are they when the block is at $x = -\frac{1}{2}x_m$?

SOLUTION: The **Key Idea** here is that, because we are given the location of the block, we can easily find the spring's potential energy with $U = \frac{1}{2}kx^2$. For $x = \frac{1}{2}x_m$, we have

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}k(\tfrac{1}{2}x_m)^2 = (\tfrac{1}{2})(\tfrac{1}{4})kx_m^2.$$

We can substitute for k and x_m, or we can use the **Key Idea** that the total mechanical energy, which we know from part (a), is $\frac{1}{2}kx_m^2$. That idea allows us to write, from the above equation,

$$U = \tfrac{1}{4}(\tfrac{1}{2}kx_m^2) = \tfrac{1}{4}E = \tfrac{1}{4}(0.393 \text{ J}) = 0.098 \text{ J}. \qquad \text{(Answer)}$$

Now, using the **Key Idea** of (a) (namely, $E = K + U$), we can write

$$K = E - U = 0.393 \text{ J} - 0.098 \text{ J} \approx 0.30 \text{ J}. \qquad \text{(Answer)}$$

By repeating these calculations for $x = -\frac{1}{2}x_m$, we would find the same answers for that displacement, consistent with the left–right symmetry of Fig. 16-6*b*.

Example 1.22. *A uniform spring of force constant k is cut into two pieces of equal length. What is the force constant of each piece ?*

Suppose, Force $= F$

Increase in length $= l$

$$k = \frac{F}{l} \qquad \ldots (i)$$

In the second, the length is half and increase in length for the force $F = \frac{l}{2}$

$$\therefore \quad k_1 = \frac{F}{l/2}$$

$$k_1 = 2\left[\frac{F}{l}\right]$$

$$k_1 = 2\,\mathbf{k} \qquad \ldots (ii)$$

Force constant for each piece = **2 k.**

Example 1.23. *A uniform spring of force constant k is cut into two pieces whose lengths are in the ratio of 1 : 3. Calculate the force constant of each piece.*

Here Force $= F$

Increase in length $= l$

$$k = \frac{F}{l} \qquad \ldots (i)$$

When the springs are cut in the ratio 1 : 3,

Increase in length for the first piece for force $F = \frac{l}{4}$

Increase in length for the second piece for force $F = \frac{3l}{4}$

$$\therefore \quad k_1 = \frac{F}{l/4} = 4\left[\frac{F}{l}\right]$$

$$k_1 = 4\,k \qquad \ldots (ii)$$

and $$k_2 = \frac{F}{3l/4} = \frac{4}{3}\left[\frac{F}{l}\right]$$

$$k_2 = \ldots \qquad \ldots (iii)$$

Example 1.24. *Two identical springs each of force constant k are connected as shown in Fig 1.15. Find the force constant of the system.*

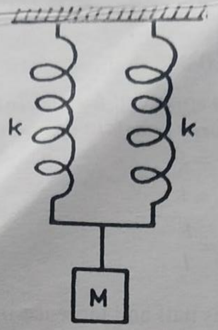


Fig. 1.15.

When force F is applied to each spring separately, increase in length $= l$

$$\therefore \quad k = \frac{F}{l} \qquad \ldots (i)$$

When force F is applied to the system, force $F/2$ acts on each spring and increase in length of the system $= \frac{l}{2}$

Force constant of the system.

$$k_1 = \frac{F}{l/2} = 2\left[\frac{F}{l}\right]$$

$$k_1 = 2\,\mathbf{k}.$$

Example 1.25. *Two identical springs each of force constant k are connected as shown in Fig. 1.16. Calculate the force constant of the system.*

When force F is applied to each spring separately, increase in length $= l$

$$\therefore \quad k = \frac{F}{l} \qquad \ldots (i)$$

When force F is applied to the system, each spring extends by l and increase in length of the system $= 2\,l$

$$\therefore \quad k_1 = \frac{F}{2l} = \frac{1}{2}\left[\frac{F}{l}\right]$$

$$k_1 = \frac{\mathbf{k}}{\mathbf{2}} \qquad \ldots (ii)$$

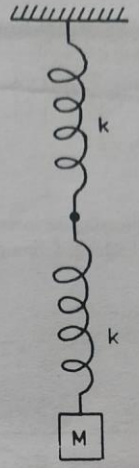


Fig. 1.16.

Example 1.26. *Two identical springs each of force constant k are connected as shown in Fig. 1.17. Find the force constant of the system.*

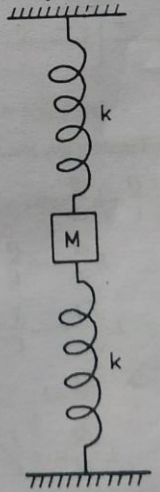


Fig. 1.17.

When force F is applied to each spring separately, increase in length $= l$

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$$k = \frac{F}{l} \qquad \ldots (i)$$

When force F is applied to the system, change in length in each spring is $l/2$. Therefore change in length of the system $= l/2$.

$$\therefore \qquad k_1 = \frac{F}{l/2} = 2\left[\frac{F}{l}\right]$$

$$k_1 = 2\,\mathbf{k}.$$