Waves and Oscillations

Lecture No. 4

Topics: Two-body oscillation, Reduced Mass, Principle of superposition of SHMs'

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Review of some earlier topics of the Course

- Differential equation of a simple harmonic oscillator
- Total energy and average energy
- Spring- mass system
- Calculation of time period of torsional pendulum
- Two-body oscillations, Reduced mass
- Combination of simple harmonic oscillations
- Lissajous figures.

Two-Body Oscillation

- In microscopic world, many objects such as nuclei, atoms, molecules, etc. execute oscillations that are approximately SHM.

- Example: Diatomic molecule in which 2 atoms are bonded together with a force. Above absolute zero temperature, the atoms vibrate continuously about their equilibrium positions.

- We can compare such a molecule with a system where the atoms can be considered as two particles with different masses connected by a spring.

Let the molecules can be represented by two masses m_1 and m_2 connected to each other by a spring of force constant k as shown in Fig 4(a).

The motion of the system can be described in terms of the separate motions of the two particles which are located relative to the origin O by the two coordinates x_1 and x_2 in Fig. 4(a).

The relative separation $(x_1 - x_2)$ gives the length of the spring at any time.

The un-stretched length of the spring is L.

The change in length of the spring is given by,

$$x = (x_1 - x_2) - L \qquad (4.1)$$

The magnitude of the force that the spring exerts on each particle is,

$$F = kx \qquad (4.2)$$

If the spring exerts a force $-\vec{F}$ on m_1, then it exerts a force $\vec{F}$ on m_2.

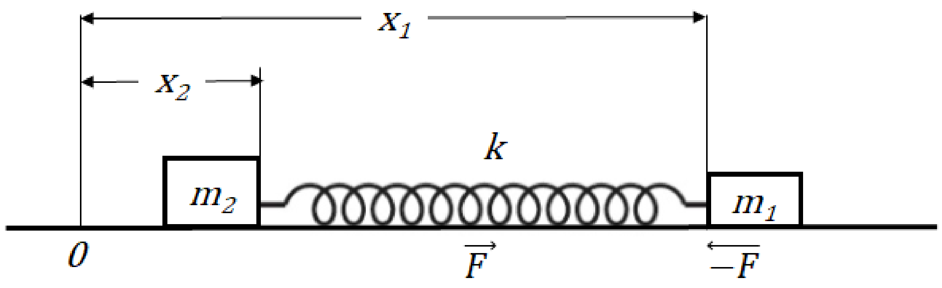

Fig. 4(a)

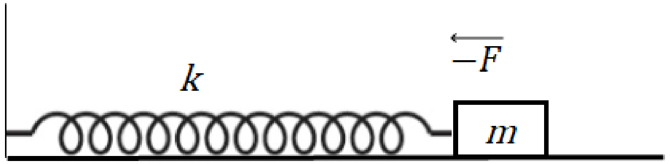

Fig. 4(b)

Taking the force component along the X-axis, let us apply Newton's 2nd law of motion separately to the two particles,

$$m_1 \frac{d^2 x_1}{dt^2} = -kx \qquad (4.3)$$

$$m_2 \frac{d^2 x_2}{dt^2} = kx \qquad (4.4)$$

Multiplying equation (4.3) by m_2 and equation (4.4) by m_1

$$m_1 m_2 \frac{d^2 x_1}{dt^2} = -m_2 kx \qquad (4.5)$$

$$m_1 m_2 \frac{d^2 x_2}{dt^2} = m_1 kx \qquad (4.6)$$

Subtracting, equation (4.6) from equation (4.5),

$$m_1 m_2 \frac{d^2 x_1}{dt^2} - m_1 m_2 \frac{d^2 x_2}{dt^2} = -m_2 kx - m_1 kx$$

$$\Rightarrow m_1\ m_2 \frac{d^2}{dt^2}(x_1 - x_2)\ = -kx(m_1 + m_2)$$

$$\Rightarrow \frac{m_1 m_2}{(m_1 + m_2)} \frac{d^2}{dt^2}(x_1 - x_2)\ = -kx \qquad (4.7)$$

The quantity $\frac{m_1 m_2}{(m_1 + m_2)}$ has the dimension of mass. This quantity is known as the reduced mass of the system and it is denoted by μ.

$$\mu = \frac{m_1 m_2}{(m_1 + m_2)} \qquad (4.8)$$

Reduced mass of a system is always smaller than either of the masses of the system. ($\mu < m_1$ and $\mu < m_2$)

Since, the un-stretched length of the spring is constant the derivative of (x_1-x_2) are the same as the derivative of x.

$$\frac{d}{dt}(x_1 - x_2) = \frac{d}{dt}(x - L) = \frac{dx}{dt} \text{ [from equation (4.1)]}$$

So, from equation (4.7) we get,

$$\mu \frac{d^2 x}{dt^2} = -kx$$

$$\Rightarrow \frac{d^2 x}{dt^2} + \frac{k}{\mu} x = 0 \qquad (4.9)$$

Here, angular frequency is, $\omega = \sqrt{\frac{k}{\mu}}$; So, time period, $T=2\pi\sqrt{\frac{\mu}{k}}$

Equation (4.9) is identical to the differential equation of SHM of a single body oscillator.

The Superposition of Oscillatory Motion

- Many Physical situations involves the simultaneous application of 2 or more periodic oscillations to the same system.

- Examples: A photograph stylus, a microphone diaphragm, a human eardrum, etc. are generally subjected to a complicated combination of combinations of many periodic oscillations.

- The basic assumption in assessment of such conditions is:

"The resultant of two or more harmonic vibrations will be taken to be simply the sum of the individual vibrations."

Principle of Superposition

Suppose we have two simple harmonic motions (SHM) described by the following equations:

$$y_1 = a_1 \sin(\omega t + \alpha_1) \qquad (4.10)$$

$$y_2 = a_2 \sin(\omega t + \alpha_2) \qquad (4.12)$$

Here, y_1 and y_2 are the displacements of the particles due to the individual vibrations of amplitudes a_1 and a_2, respectively and angles of epoch $\alpha 1$ and α_2, respectively. Two vibrations have same the angular frequency (ω).

The resultant displacement of the particle will be given by,

$$y = y_1 + y_2$$

$$= a_1 \sin(\omega t + \alpha_1) + a_2 \sin(\omega t + \alpha_2)$$

$$= a_1(\sin \omega t \cos \alpha_1 + \cos \omega t \sin \alpha_1) + a_2(\sin \omega t \cos \alpha_2 + \cos \omega t \sin \alpha_2)$$

$$= (a_1 \cos \alpha_1 + a_2 \cos \alpha_2) \sin \omega t + (a_1 \sin \alpha_1 + a_2 \sin \alpha_2) \cos \omega t \qquad (4.13)$$

Since, a_1, a_2 (amplitudes) and $\alpha 1$ and α_2 (angle of epoch) are constants we can replace them with the following constant terms:

$$a_1 \cos \alpha_1 + a_2 \cos \alpha_2 = A \cos \varphi$$

$$a_1 \sin \alpha_1 + a_2 \sin \alpha_2 = A \sin \varphi$$

The resultant amplitude can be written as,

$$y = A\cos\varphi\sin\omega t + A\sin\varphi\cos\omega t = A\sin(\omega t + \varphi) \qquad (4.14)$$

Thus the equation (4.14) is similar to the equations (4.11) and (4.12). The resultant vibration is therefore represents a SHM with the amplitude A and epoch angle φ.

Expression of A and φ:

$$A^2\sin^2\varphi + A^2\cos^2\varphi = a_1^2 \sin^2 \alpha_1 + a_2^2 \sin^2 \alpha_2 + 2a_1a_2\sin\alpha_1\sin\alpha_2 + a_1^2 \cos^2 \alpha_1 + a_2^2 \cos^2 \alpha_2 + 2a_1a_2\cos\alpha_1\cos\alpha_2$$

Or, $A^2(\sin^2\varphi + \cos^2\varphi) = a_1^2 (\sin^2 \alpha_1 + \cos^2\alpha_1) + a_2^2(\sin^2\alpha_2 + \cos^2\alpha_2) + 2a_1a_2(\sin\alpha_1\sin\alpha_2 + \cos\alpha_1\cos\alpha_2)$

Or, $A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\alpha_1 - \alpha_2)$

Or, $A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos(\alpha_1 - \alpha_2)}$ (4.15)

$\tan \varphi = \frac{A\sin\varphi}{A\cos\varphi} = \frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2}$ (4.16)

Some Special Cases:

i. Same Phase: If $\alpha_1 = \alpha_2 = \alpha, \alpha_1 = \alpha_2 = 0, 2\pi, 4\pi, \dots\dots\dots = 2n\pi$; $n = 0, 1, 2, 3, \dots\dots\dots\dots\dots$

Then, $\cos(\alpha_1 - \alpha_2) = 1$ and $A^2 = a_1^2 + a_2^2 + 2a_1a_2 = (a_1 + a_2)^2$; So, $A = (a_1 + a_2)$

$$tan\varphi = \frac{(a_1+a_2)sin\alpha}{(a_1+a_2)cos\alpha} = tan\alpha$$

In this case, $y = (a_1 + a_2)\sin(\omega t + \alpha)$ (4.17)

ii. Opposite Phase: If $\alpha_1 - \alpha_2 = \pi, 3\pi, 5\pi, \dots\dots\dots = (2n + 1)\pi$; $n = 0, 1, 2, 3, \dots\dots\dots\dots\dots$

Then, $\cos(\alpha_1 - \alpha_2) = -1$ and $A^2 = a_1^2 + a_2^2 - 2a_1a_2 = (a_1 - a_2)^2$; So, $A = (a_1 - a_2)$

iii. If $a_1 = a_2 = a$; The same phase condition gives, $A = 2a$; Resultant amplitude is the maximum.

The opposite phase condition gives $A = 0$; Resultant amplitude is zero.

iv. If $\alpha_1 - \alpha_2 = \frac{\pi}{2}$; $A = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos\frac{\pi}{2}} = \sqrt{a_1^2 + a_2^2}$

In this case if $a_1 = a_2 = a$, $A = \sqrt{2a^2} = \sqrt{2}a$